

Triangle groups, automorphic forms, and torus knots

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Abstract: This article is concerned with the relation between several classical and well-known objects: triangle Fuchsian groups, \mathbb{C}^\times -equivariant singularities of plane curves, torus knot complements in the 3-sphere. Torus knots are the only nontrivial knots whose complements admit transitive Lie group actions. In fact $\mathbb{S}^3 \setminus K_{p,q}$ is diffeomorphic to a coset space $\widetilde{SL}_2(\mathbb{R})/G$, where $\widetilde{SL}_2(\mathbb{R})$ is the universal covering group of $PSL_2(\mathbb{R})$ and G is a discrete subgroup contained in the preimage of a (p, q, ∞) -triangle Fuchsian group. The existence of a diffeomorphism between $\widetilde{SL}_2(\mathbb{R})/G$ and $\mathbb{S}^3 \setminus K_{p,q}$ is known from a general topological classification of Seifert fibred 3-manifolds. Our goal is to construct an explicit diffeomorphism using automorphic forms. Such a construction is previously known for the trefoil knot $K_{2,3}$ and in fact $\mathbb{S}^3 \setminus K_{2,3} \cong SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. The connection between the two sides of the diffeomorphism comes via singularities of plane curves.

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1 Introduction

The aim of this paper is to construct an explicit diffeomorphism between the complement $\mathbb{S}^3 \setminus K_{p,q}$ of a (p,q) -torus knot in the 3-sphere and a coset space $\widetilde{SL}_2(\mathbb{R})/G$ of the simply connected Lie group $\widetilde{SL}_2(\mathbb{R})$ with respect to a suitable discrete subgroup G . The fact that $\mathbb{S}^3 \setminus K_{p,q}$ is diffeomorphic to a coset space of $\widetilde{SL}_2(\mathbb{R})$ is known, it follows from a general classification of Seifert manifolds, cf [12]. We present a proof using automorphic forms for triangle groups, in the spirit of [9].

Since $\widetilde{SL}_2(\mathbb{R})$ is a model for one of the eight 3-dimensional geometries in Thurston's list, the results presented here can be regarded as a construction of a geometric structure on torus knot complements. Let us briefly recall some facts in this context, the necessary definitions can be found in the background section below. Papakyriakopoulos proved in the 50's that knot complements are aspherical and the universal cover of $\mathbb{S}^3 \setminus K$ is homeomorphic to \mathbb{R}^3 , for any knot K . The complement of a trivial knot is homeomorphic to a solid torus and admits geometric structures modeled on all of Thurston's geometries, except \mathbb{S}^3 and $\mathbb{S}^2 \times \mathbb{E}$. Thurston proved that if the complement $\mathbb{S}^3 \setminus K$ of a nontrivial knot admits a geometric structure, then this structure is hyperbolic if and only if K is not a torus knot, see [15]. A knot whose complement admits a hyperbolic structure is called a hyperbolic knot; its group is represented as a Kleinian group. We are not concerned with hyperbolic knots in this paper. As for torus knots, the most important topological features of their complements where described in the classical paper of Seifert, [18]. In particular, it is shown there that $\mathbb{S}^3 \setminus K_{p,q}$ admits a Seifert fibration. The base orbifold Θ of this Seifert fibration is a punctured sphere with two cone points. It is not hard to see that Θ can be obtained as the quotient of the hyperbolic plane \mathbb{H}^2 under the action of a (p, q, ∞) -triangle Fuchsian group $\Gamma_{p,q}$. This suggests that the only possible geometries for torus knot complements are $\mathbb{H}^2 \times \mathbb{E}$ and $\widetilde{SL}_2(\mathbb{R})$. In fact, both structures exist on $\mathbb{S}^3 \setminus K_{p,q}$. The standard approach for the construction of a $\mathbb{H}^2 \times \mathbb{E}$ structure uses another feature of torus knot complements, which was also observed by Seifert. Namely, the fact that $\mathbb{S}^3 \setminus K_{p,q}$ fibres over \mathbb{S}^1 , with fibre - a smooth open surface, and a periodic monodromy map. These properties imply the existence of a $\mathbb{H}^2 \times \mathbb{E}$ structure, see e.g. [17]. As mentioned above, the presence of a geometric structure modeled on $\widetilde{SL}_2(\mathbb{R})$ is shown by Raymond and Vasquez, [12], who used the theory of Seifert fibrations to give a topological classification of 3-manifolds universally covered by Lie groups. The dichotomy between $\mathbb{H}^2 \times \mathbb{E}$ and $\widetilde{SL}_2(\mathbb{R})$ is discussed in some detail in Bonahon's survey [3].

Raymond and Vasquez noticed that the only torus knot whose complement can be obtained as a coset of the simple Lie group $PSL_2(\mathbb{R})$ is the trefoil $K_{2,3}$. In all other cases the group G is not the full preimage of a Fuchsian group and the coset $\widetilde{SL}_2(\mathbb{R})/G$ cannot be reduced to a coset of $PSL_2(\mathbb{R})$. In the case of the trefoil, we have $\mathbb{S}^3 \setminus K_{2,3} \cong PSL_2(\mathbb{R})/PSL_2(\mathbb{Z}) \cong SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, so G is the full preimage of the modular group. This curious fact has an interesting analytic proof due to Quillen, see [8] §10.

Since we intend to generalize a version of Quillen's argument, it is suitable to present

the idea here. Recall that the algebra of modular forms for $\Gamma = SL_2(\mathbb{Z})$ is generated by two elements, often denoted g_2, g_3 . The modular form $\Delta = g_2^3 - 27g_3^2$ is the cusp form of least possible degree, vanishes with order 1 at ∞ and doesn't vanish anywhere in the upper half plane \mathbb{H}^2 . Let $T'\mathbb{H}^2$ denote the tangent bundle of \mathbb{H}^2 with removed zero section, and $U\mathbb{H}^2$ denote the unit tangent bundle. Classically, modular forms are defined as functions on \mathbb{H}^2 with some specific behavior under the action of Γ . However, it is well known that modular forms can be regarded as specific functions on $T'\mathbb{H}^2$ invariant on the orbits of (the tangent action of) Γ . Now define a map $\Psi : T'\mathbb{H}^2 \rightarrow \mathbb{C}^2$ sending v to $(g_2(v), g_3(v))$. Since the modular forms are constant on Γ -orbits and Δ is nowhere zero, Ψ factors through a map $\overline{\Psi} : T'\mathbb{H}^2/\Gamma \rightarrow \mathbb{C}^2 \setminus V$ where $V = \{z_1^3 - 27z_2^2 = 0\}$. It is not hard to show that $\overline{\Psi}$ is biholomorphic. Now notice that $PSL_2(\mathbb{R})$ acts simply transitively on $U\mathbb{H}^2$, hence the two are diffeomorphic and furthermore $U\mathbb{H}^2/\Gamma \cong PSL_2(\mathbb{R})/\Gamma$. Thus $PSL_2(\mathbb{R})/\Gamma$ embeds into $\mathbb{C}^2 \setminus V$. On the other hand, if \mathbb{S}^3 is the unit sphere in \mathbb{C}^2 , the intersection $K = \mathbb{S}^3 \cap V$ is a trefoil knot. It is not hard to see that $\overline{\Psi}(U\mathbb{H}^2/\Gamma)$ and $\mathbb{S}^3 \setminus K$ are related by an isotopy in \mathbb{C}^2 , and this completes the argument. Formally, Quillen regards $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ as a parameter space for unimodular lattices in \mathbb{R}^2 , and g_2, g_3 as invariants of such lattices.

Our goal is to generalize the construction sketched above to incorporate all (p, q, ∞) -triangle groups and, respectively, all torus knots. Here is a brief outline of the paper and the main results. Section 2 contains some background on geometric structures, orbifolds, Seifert fibrations, and knots. In particular, we recall that a torus knot $K_{p,q}$ is obtained as the intersection of \mathbb{S}^3 with a singular plane curve of the form $\{c_1z_1^q + c_2z_2^p = 0\}$. We discuss the \mathbb{C}^\times action for which V is equivariant and the Seifert fibration in $\mathbb{S}^3 \setminus K$ given by the restriction of this action to \mathbb{S}^1 . Section 3 contains some basic material on $\widetilde{SL}_2(\mathbb{R})$, its action on the hyperbolic plane, its discrete subgroups, and automorphic forms. We consider automorphic forms of fractional degrees, whose transformation laws involve characters of the discrete group in question. Such automorphic forms are defined as functions on the universal cover $\widetilde{T'\mathbb{H}^2}$ of $T'\mathbb{H}^2$, a space on which $\widetilde{SL}_2(\mathbb{R})$ acts without stabilizers having $\widetilde{U\mathbb{H}^2}$ as a representative orbit. Section 4, the core of the paper, is devoted to (p, q, ∞) -triangle groups. We start with the definition and elementary properties of $\Gamma_{p,q}$. We observe that the preimage $\widetilde{\Gamma}$ of $\Gamma_{p,q}$ in $\widetilde{SL}_2(\mathbb{R})$ is abstractly isomorphic to the torus knot group $G_{p,q} = \pi_1(\mathbb{S}^3 \setminus K_{p,q})$. In the last paragraph of this section we study the algebra of automorphic forms for $\widetilde{\Gamma}$. In theorem 4.14 we determine generators for the algebra of automorphic forms, denoted ω_a, ω_b . The cusp form of lowest degree is written as a linear combination $\omega_\infty = c_a\omega_a^p + c_b\omega_b^q$, and does not vanish on $\widetilde{T'\mathbb{H}^2}$. We also show that the characters of $\widetilde{\Gamma}$ associated with ω_a and ω_b share a kernel, G , which has index $pq - p - q$ in $\widetilde{\Gamma}$. Thus the two generators are constant on the orbits of G in $\widetilde{T'\mathbb{H}^2}$. Section 5 is the culmination of the paper. We use the generators ω_a, ω_b to define a map from $\widetilde{T'\mathbb{H}^2}$ to \mathbb{C}^2 . In theorem 5.1 we show that this map factors through a biholomorphic map $\widetilde{T'\mathbb{H}^2}/G \rightarrow \mathbb{C}^2 \setminus V$ where $V = \{c_bz_1^q + c_az_2^p = 0\}$. The intersection $K = \mathbb{S}^3 \cap V$ is a (p, q) -torus knot. The complement $\mathbb{S}^3 \setminus K$ is related to $\overline{\Psi}(\widetilde{U\mathbb{H}^2}/G)$ via an isotopy of \mathbb{C}^2 . In the last section 6 we

identify the coset space $PSL_2(\mathbb{R})/\Gamma_{p,q}$ as a knot complement in a lens space.

The paper contains a fair amount of known material, especially about torus knot groups, included in order to reduce the necessary prerequisites. As far as automorphic forms are concerned, some definitions and ideas are taken from Milnor's paper [9], where the case of co-compact triangle groups is considered. Another very important reference is Ogg's book [11]. Ogg works in the classical setting where the automorphic forms are defined as functions on \mathbb{H}^2 , fractional degrees and characters are allowed but $\widetilde{SL}_2(\mathbb{R})$ is not mentioned explicitly. The first chapter of this book contains a treatment of $(2, q, \infty)$ -triangle groups, and a construction for the generators of the algebra of automorphic forms.

Other sources relating automorphic forms and quasi-homogeneous singularities include the following. Dolgachev, [5], gives an outline of the relation just mentioned, in the case of co-compact Fuchsian groups, with an emphasis on special triangle groups. The same author, in [6], outlines a generalization of these ideas, replacing the hyperbolic plane with a higher dimensional complex homogeneous space. Wagreich, [19], [20], [21], considers Fuchsian groups of the first kind, not necessarily co-compact. In particular, Wagreich provides a classification of all such groups whose algebra of automorphic forms admits a generating set of at most 3 elements. (This result does not cover our case because Wagreich considers only forms of integral degree, while here we need to allow fractional degrees.) More recently, Natanson and Pratoussevitch, [10], have used and developed this framework in their study of moduli spaces of Gorenstein singularities.

2 Background

2.1 Geometric structures

A complete and locally homogenous Riemannian metric g on a manifold M is called a **geometric structure** on M . Here complete means that every geodesic in M may be extended to $(-\infty, \infty)$; locally homogenous means that any two points in M have isometric neighborhoods. A complete, homogenous, simply connected Riemannian manifold (N, h) , whose isometry group is maximal, is called a **geometry**. We shall say that the geometric structure (M, g) is modeled on the geometry (N, h) , if each point of M has a neighborhood, isometric to an open set of N .

Let M be a manifold and let \widetilde{M} be the universal cover of M . If \widetilde{M} admits a geometry g such that the covering action of $\pi_1(M)$ is by isometries, the covering map induces a geometric structure on M modeled on (\widetilde{M}, g) .

In dimension 2, the uniformization theorem of Poincaré states, that there are three geometries: the Euclidian plane, the hyperbolic plane and the sphere:

$$\mathbb{E}^2, \quad \mathbb{H}^2, \quad \mathbb{S}^2.$$

Any 2-manifold has a geometric structure obtained by identifying its universal cover with one of the three geometries above. Some 2-manifolds (\mathbb{R}^2 , $\mathbb{R}^2 \setminus \{\text{point}\}$, Möbius band) have geometric structures modeled on two distinct geometries.

The situation with 3-manifolds is much more complicated. Thurston, [16], showed that the 3-dimensional geometric structures are to be modeled on of the following eight geometries:

$$\begin{array}{lll} \mathbb{E}^3 & \mathbb{S}^2 \times \mathbb{E} & \text{Nil} \\ \mathbb{H}^3 & \mathbb{H}^2 \times \mathbb{E} & \text{Sol} \\ \mathbb{S}^3 & \widetilde{SL}_2(\mathbb{R}) & \end{array} \quad (1)$$

However, it is not true that any 3-manifold readily admits a geometric structure. The theory was recently lead to a great success, see [14], but we shall refrain from comments on the general results, as our concerns here are modest in this respect and we only consider manifolds whose geometry is fairly well-known. We refer the reader to the surveys of Scott [13] and Bonahon [3], which predate the general existence theorem, but contain excellent descriptions of the eight geometries, as well as many examples. The definitions and results needed for our purposes are stated in the text below.

2.2 Seifert fibrations and orbifolds

In this section we discuss briefly a class of 3-manifolds introduced by Seifert [18]. These are manifolds admitting a so called Seifert fibration - a special kind of circle foliation. The "base" has an orbifold structure as described below. The Seifert fibration structures on 3-manifolds are closely related to their geometric properties, as explained in Scott's article [13], which we follow here to some extend.

We start with the general definition of an orbifold as introduced by Thurston in [15]. Intuitively, an orbifold is a space locally modeled on \mathbb{R}^n modulo finite group actions.

Definition 2.1. *An n -dimensional **orbifold** Θ consists of the following data. A Hausdorff paracompact topological space X_Θ , called the **underlying space**, covered by a collection $\{U_i\}$ of open sets, called **charts**, closed under finite intersections. To each U_i is associated a finite group Γ_i , an action of Γ_i on an open subset \tilde{U}_i of \mathbb{R}^n , and a homeomorphism $\varphi_i : U_i \rightarrow \tilde{U}_i/\Gamma_i$. Whenever $U_i \subset U_j$ there is an injective homomorphism*

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and an embedding

$$\tilde{\varphi}_{ij} : \tilde{U}_i \hookrightarrow \tilde{U}_j$$

equivariant with respect to f_{ij} (i.e., for $\gamma \in \Gamma_i$, $\tilde{\varphi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\varphi}_{ij}(x)$) such that the

diagram below commutes:

$$\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{\varphi}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/\Gamma_i & \xrightarrow{\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i} & \tilde{U}_j/\Gamma_j \\
\uparrow \varphi_i & & \downarrow f_{ij} \\
U_i & \hookrightarrow & U_j
\end{array}$$

If $x \in X_\Theta$ and $U = \tilde{U}/\Gamma$ is a chart about x , we denote by Γ_x the isotropy group of any point in the preimage of x in \tilde{U} . The set $\Sigma_\Theta := \{x \in X_\Theta : \Gamma_x \neq \{1\}\}$ is called the **singular locus** of Θ . The points of Σ_Θ are called **singular**, the rest of the points of X_Θ - **regular**.

Note, that the maps $\tilde{\varphi}_{ij}$ are defined up to composition with elements of Γ_j , and f_{ij} are defined up to conjugation by elements of Γ_j . It is not generally true that $\tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ when $U_i \subset U_j \subset U_k$, but there should exist an element $\gamma \in \Gamma_k$ such that $\gamma \tilde{\varphi}_{ik} = \tilde{\varphi}_{jk} \circ \tilde{\varphi}_{ij}$ and $\gamma f_{ik}(g)\gamma^{-1} = f_{jk} \circ f_{ij}(g)$. Another remark to be made is that the covering $\{U_i\}$ is not an intrinsic part of the structure of the orbifold: two coverings give rise to the same orbifold structure if they can be combined consistently to give a finer cover still satisfying the above conditions.

Note also, that any manifold is an orbifold with empty singular locus.

Definition 2.2. An **orbifold-cover** of an orbifold Θ is an orbifold Ξ together with a projection $\rho : X_\Xi \longrightarrow X_\Theta$, such that every point $x \in X_\Theta$ has a neighborhood $U = \tilde{U}/\Gamma$ such that each component V_i of $\rho^{-1}(U)$ is isomorphic to \tilde{U}/Γ_i , where Γ_i is a subgroup of Γ . The isomorphism $\rho^{-1}(U) \cong \tilde{U}/\Gamma_i$ is required to respect the projections. We use the notation $\rho : \Xi \longrightarrow \Theta$.

An orbifold-cover $\rho : \tilde{\Theta} \longrightarrow \Theta$ is called **universal**, if for any other orbifold-cover $\rho : \Xi \longrightarrow \Theta$, there is a lift $\tilde{\rho} : \tilde{\Theta} \longrightarrow \Xi$ which is an orbifold-cover, and $\rho = \tilde{\rho} \circ \rho$.

An orbifold is called **good** if it admits an orbifold-cover which is in fact a manifold, and **bad** - otherwise.

Proposition 2.1. (Thurston [15]) Let M be a manifold and let G be a group acting properly discontinuously on M . Then M/G has the structure of an orbifold and the projection $M \longrightarrow M/G$ is an orbifold-cover.

In the above notation, if H is a normal subgroup of G , then M/H is an orbifold-cover of M/G under the action of the factorgroup G/H . If M is a simply connected manifold, it is the universal orbifold-cover of M/G . Existence of a universal orbifold-cover $\tilde{\Theta}$ for an arbitrary orbifold Θ is shown in [15], along with the description of the corresponding

group of deck transformations called the **orbifold fundamental group** and denoted $\pi_1(\Theta)$.

In the 2-dimensional case, both the universal cover and the fundamental group of an orbifold are easier to describe. An outline can be found in [13]. If Θ is a good 2-dimensional orbifold, covered by a manifold M , then we can define a geometric structure on Θ , using the geometry of the universal cover \tilde{M} of M , which must be $\mathbb{E}^2, \mathbb{S}^2$ or \mathbb{H}^2 .

From now on we consider only 2-dimensional orbifolds, and we restrict the type of allowed singularities to cone points, which are defined as follows. A singular point $x \in X_\Theta$ is called a **cone point** of index k , if there is a chart U about x such that the corresponding group Γ_x is isomorphic to \mathbb{Z}_k and acts by rotations around a point $\tilde{x} \in \tilde{U}$. Note that if Θ is an orbifold whose singular locus contains only cone points, then the singularities of any orbifold-cover of Θ are also limited to cone points. Scott, in [13], gives the following method to compute the fundamental group of a 2-dimensional orbifold with cone points.

Proposition 2.2. *Let Θ be a 2-dimensional orbifold with singular locus consisting of a finite number of cone points a_1, \dots, a_n with indexes k_1, \dots, k_n respectively. Let D_1, \dots, D_n be disjoint disc neighborhoods of a_1, \dots, a_n . We set $N := X_\Theta \setminus (D_1 \cup \dots \cup D_n)$. Let H be the smallest normal subgroup of $\pi_1(N)$ containing the elements $c_1^{k_1}, \dots, c_n^{k_n}$, with c_j representing the circle ∂D_j . Then*

$$\pi_1(\Theta) \cong \pi_1(N)/H$$

Suppose $\pi_1(N)$ has a presentation with generators x_1, \dots, x_l and relations $r_1 = 1, \dots, r_m = 1$ where r_j is some word in the generators. Then to obtain a presentation for $\pi_1(\Theta)$ is sufficient to express c_1, \dots, c_n in terms of x_1, \dots, x_l and one gets

$$\pi_1(\Theta) = \langle x_1, \dots, x_l ; r_1 = 1, \dots, r_m = 1, c_1^{k_1} = 1, \dots, c_n^{k_n} = 1 \rangle .$$

We are now ready to proceed with the definition of Seifert fibred 3-manifolds, and to describe some of their basic properties.

Definition 2.3. *The solid torus $T := \mathbb{D}^2 \times \mathbb{S}^1$ is a trivial circle bundle over the disc $\mathbb{D}^2 := \{z \in \mathbb{C} : |z| = 1\}$. The fibres are $(z, e^{2\pi it})$, $t \in [0, 1]$. The torus T with this fibration structure is called **trivially fibred torus**.*

Let p and q be co-prime integers. The solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ with the circle foliation

$$(ze^{2\pi i \frac{p}{q}t}, e^{2\pi it}) , \quad t \in [0, q]$$

is called **(p,q)-twisted fibred torus**, and denoted by $T(p, q)$. The central fibre $\{0\} \times \mathbb{S}^1$ of a fibred torus is called the **core**.

Definition 2.4. *A 3-manifold M is called **Seifert manifold**, if it can be presented as a disjoint union of circles, called fibres, satisfying the following property. Each fibre l admits a tubular neighborhood T_l in M , consisting of fibres, such that T_l is a fibred torus (possibly trivial) with core l .*

If a given fibre possesses a neighborhood which is a trivial fibred torus, the fibre is called **regular**. Otherwise the fibre is called **singular**.

A fibration of this kind is called a **Seifert fibration** on M . The fibred tori T_l are called **trivializing tori** of the fibration.

A fibred torus $T(p, q)$ can be covered by a trivially fibred torus in different ways. Each of the following transformations generates an action of the cyclic group \mathbb{Z}_q on $\mathbb{D}^2 \times \mathbb{S}^1$, which sends fibres to fibres.

$$\begin{aligned} \mathbb{D}^2 \times \mathbb{S}^1 &\longrightarrow \mathbb{D}^2 \times \mathbb{S}^1 \\ (z, e^{2\pi it}) &\longmapsto (ze^{2\pi i \frac{p}{q} t}, e^{2\pi it}) \end{aligned} \quad (2)$$

$$\begin{aligned} \mathbb{D}^2 \times \mathbb{S}^1 &\longrightarrow \mathbb{D}^2 \times \mathbb{S}^1 \\ (z, e^{2\pi it}) &\longmapsto (ze^{2\pi i \frac{p}{q} t}, e^{2\pi i(t + \frac{1}{q})}) \end{aligned} \quad (3)$$

The first transformation keeps all points in the core fixed, and its action results in an orbifold-covering between the solid tori. The second one defines a regular covering of manifolds.

The set of fibres in a fibred torus form an orbifold described as follows. In the notation of the definition of a fibred torus, put $D = \mathbb{D}^2 \times \{1\} \subset T$. If T is trivially fibred, then each fibre intersects D exactly once, and hence D parametrizes the fibres. In this case we have a regular fibration, and the base space is the manifold D . If T is (p, q) -fibred, then the core intersects D once, while each of the other fibres intersects D at q distinct points. The set of fibres is the orbifold D/\mathbb{Z}_q , where \mathbb{Z}_q is viewed as the group of q -th roots of 1 acting on D by multiplication; the singular locus consists of a single cone point. More generally, the set of fibres in a Seifert fibred 3-manifold M is a 2-dimensional orbifold Θ whose singular locus consists of cone points corresponding to the singular fibres in M . In such case Θ is called the **base orbifold** of the Seifert fibration M . When the singular fibres are finite in number, we can compute the fundamental group of the base orbifold using proposition 2.2.

Remark 2.1. Let M be a connected Seifert fibred manifold.

(*) All regular fibres in M are isotopic. Indeed, let l and l' be two regular fibres. Let γ be a path connecting l with l' , which does not intersect a singular fibre. Then γ can be covered by a finite number of open trivial fibred solid tori T_1, \dots, T_n , such that $T_1 \supset l$ and $T_n \supset l'$. Now the result follows from the fact, that all fibres in a trivial fibred solid torus are isotopic.

(**) Let l be a regular fibre, let m be a base point for $\pi_1(M)$ lying on a regular fibre, and let γ be a path from the point m to l . Then the element $\tau \in \pi_1(M, m)$ represented by $\gamma^{-1} \cdot l \cdot \gamma$ does not depend on the choice of γ . Indeed, if γ_1 is another path from m to l and l_m be the regular fibre containing m , then, by (*) above, $\gamma^{-1} \cdot l \cdot \gamma \sim l_m \sim \gamma^{-1} \cdot l \cdot \gamma$, where \sim denotes a homotopy keeping the base point m fixed.

Hence, the subgroup of $\pi_1(M, m)$ generated by τ is normal, and does not depend on the choice of m and l . We refer to this subgroup as **the subgroup of $\pi_1(M)$ generated by a regular fibre**.

Lemma 2.3. *Let M be a connected Seifert fibred manifold with base orbifold Θ . Let L be the subgroup of $\pi_1(M)$ generated by a regular fibre. Then L is contained in the center of $\pi_1(M)$ and there is an exact sequence*

$$1 \longrightarrow L \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Theta) \longrightarrow 1 \quad (4)$$

Proof. See e.g. [13] lemma 3.2. □

Example 2.1. *Let Θ be a good 2-dimensional orbifold, and let P be its universal cover. Thus P is either \mathbb{S}^2 , \mathbb{E}^2 or \mathbb{H}^2 , and Θ is isomorphic to P/Γ for some discrete subgroup $\Gamma \subset \text{Isom}(P)$. Let UP be the unit tangent bundle on P , i.e. the bundle whose total space consists of all tangent vectors to P with unit length. Then Γ acts freely on UP and the quotient $M = UP/\Gamma$ is a smooth manifold. There is a Seifert fibration on M whose fibres are the images of the fibres $U_p P$, for $p \in P$. The base orbifold of this Seifert fibration is Θ . It is natural to call M the unit tangent bundle on Θ . The fundamental group $\pi_1(M)$ is a central extension of Γ by $\pi_1(UP)$. For the three possible cases we have $\pi_1(U\mathbb{S}^2) = \mathbb{Z}_2$, $\pi_1(U\mathbb{E}^2) = \mathbb{Z}$, $\pi_1(U\mathbb{H}^2) = \mathbb{Z}$.*

2.3 Torus knots

We refer to Burde and Zieschang [4] for the classical knot theoretic notions, some history and bibliography. In this section we focus on some properties of torus knots needed for our purposes.

An embedded circle $K \hookrightarrow \mathbb{S}^3$ is called a **knot**. The knot type of K is the isotopy equivalence class of the embedding. We only consider tame (smooth) knots; for such knots K and L the relation of isotopy is equivalent to the existence of an orientation preserving homeomorphism of \mathbb{S}^3 sending K to L . The 3-sphere will be considered as the unit sphere in the complex place \mathbb{C}^2 , i.e.

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\} . \quad (5)$$

A knot is called **trivial** (or **unknot**), if it is equivalent to the circle $U := \{|z_1| = 1, z_2 = 0\}$. A nontrivial knot is called a **torus knot** if it is equivalent to a simple closed curve on the torus

$$T^2 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = a, |z_2| = b\} \subset \mathbb{S}^3 \quad (6)$$

where a and b are positive numbers satisfying $a^2 + b^2 = 1$. The isotopy class of such a curve K is determined by the pair of co-prime integers (p, q) , which represent K as an element of the fundamental group of T^2 in terms of the standard generators $\{|z_1| = a, z_2 = b\}$

and $\{z_1 = a, |z_2| = b\}$. In such a case K is called a (p, q) -torus and is denoted by $K_{p,q}$. We have the relation $K_{p,q} \sim K_{p',q'}$ if and only if (p', q') equals $(-p, -q)$, (q, p) or $(-q, -p)$. The knot $K_{p,-q}$ is the mirror image of $K_{p,q}$; the complements $\mathbb{S}^3 \setminus K_{p,q}$ and $\mathbb{S}^3 \setminus K_{p,-q}$ are homeomorphic but not under an orientation preserving homeomorphism.

The fundamental group $\pi_1(\mathbb{S}^3 \setminus K)$ is called the **knot group** and denoted by G_K . One important result characterizing torus knots, conjectured by Neuwirth and proved by Burde and Zieschang, is the following: a knot group admits a non-trivial center if and only if the knot is either an unknot or a torus knot; see [4] for details and further references. This algebraic characterization is related to the presence of a Seifert fibration in a torus knot complement, the classical result of Seifert mentioned earlier.

Let us be more explicit. Let p and q be co-prime natural numbers, fixed for the rest of the paper. Assume $p < q$. Consider the linear \mathbb{C}^\times action on \mathbb{C}^2 given by

$$\begin{aligned} \mathbb{C}^\times \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ \lambda \cdot (z_1, z_2) &= (\lambda^p z_1, \lambda^q z_2). \end{aligned}$$

The origin $\mathbf{o} := (0, 0) \in \mathbb{C}^2$ is a fixed point of the action. The quotient

$$\mathbb{P}^1(p, q) = (\mathbb{C}^2 \setminus \{\mathbf{o}\}) / \mathbb{C}^\times$$

is the so called (p, q) -weighted complex projective line. Restricting λ to \mathbb{S}^1 we obtain the unitary flow

$$h_t := \begin{pmatrix} e^{2\pi i p t} & 0 \\ 0 & e^{2\pi i q t} \end{pmatrix}, \quad t \in \mathbb{R}. \quad (7)$$

Proposition 2.4. *The action of the flow h_t defines a Seifert fibration in \mathbb{S}^3 . The base orbifold, denoted $\mathbb{P}^1(p, q)$, has underlying topological space \mathbb{S}^2 , and two singular points: cone points of indices p and q respectively. The orbifold fundamental group is trivial.*

Proof. All 3-spheres centered in \mathbf{o} are invariant under h_t . Thus h_t defines a circle foliation in \mathbb{S}^3 . To show that this foliation is actually a Seifert fibration we need to find a system of trivializing tori (see definition 2.4). The coordinate complex lines \mathbb{C}_1 and \mathbb{C}_2 in \mathbb{C}^2 are also h_t -invariant. $\mathbb{C}_1 \setminus \{\mathbf{o}\}$ and $\mathbb{C}_2 \setminus \{\mathbf{o}\}$ consist of the orbits with periods $\frac{1}{p}$ and $\frac{1}{q}$ respectively. The remaining orbits have period 1. Put

$$\begin{aligned} S_1 &= \mathbb{C}_1 \cap \mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = 1, z_2 = 0\}, \\ S_2 &= \mathbb{C}_2 \cap \mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0, |z_2| = 1\}. \end{aligned}$$

The solid torus $\mathbb{S}^3 \setminus S_1$ is h_t -invariant. The orbits of the flow are of the form $(z_1 e^{2\pi i p t}, z_2 e^{2\pi i q t})$, and by reparametrization $t' = tq$ we obtain $(z_1 e^{2\pi i \frac{p}{q} t'}, z_2 e^{2\pi i t'})$. Hence, the flow h_t defines a structure of a (p, q) -twisted fibred torus in $\mathbb{S}^3 \setminus S_1$. The core of this fibred torus is S_2 , fixed by the transformation $h_{\frac{q}{p}}$. Analogously, $\mathbb{S}^3 \setminus S_2$ is a (q, p) -twisted fibred torus, whose core S_1 is fixed by the transformation $h_{\frac{p}{q}}$. All orbits except S_1 and S_2 have period 1, and

are regular in the sense of Seifert. Thus \mathbb{S}^3 is a Seifert fibration with two singular fibres S_1 and S_2 . The base orbifold $\mathbb{P}^1(p, q)$ has two singular cone points a and b of indexes p and q respectively. The set of regular points $\mathbb{P}^1(p, q) \setminus \{a, b\}$ is an annulus, because it is the orbit space of $\mathbb{S}^3 \setminus (S_1 \cup S_2)$ which is a fibred torus with removed core. It follows that the underlying surface of $S(p, q)$ is a 2-sphere. The fact that the orbifold fundamental group of $S(p, q)$ is trivial can be obtained directly from Proposition 2.2 using the fact that p and q are co-prime. Alternatively, it follows from Lemma 2.3 and the fact that \mathbb{S}^3 is simply connected. \square

Any regular orbit of h_t is a (p, q) -torus knot in the 3-sphere in which it belongs. Explicitly, let z_{01}, z_{02} be two nonzero complex numbers, then the orbit through (z_{01}, z_{02}) is a (p, q) -knot lying on the torus

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_{01}|, |z_2| = |z_{02}|\} \subset \mathbb{S}_{|z_{01}|^2 + |z_{02}|^2}^3.$$

Now let c_1, c_2 be any two nonzero complex numbers, and consider the polynomial

$$f(z_1, z_2) := c_1 z_1^p + c_2 z_2^q.$$

The analytic set $V(f) := \{(z_1, z_2) \in \mathbb{C}^2 : c_1 z_1^q + c_2 z_2^p = 0\}$ has a unique singularity at the origin. We have

$$f(\lambda \cdot (z_1, z_2)) = \lambda^{pq} f(z_1, z_2). \quad (8)$$

Hence, $V(f)$ is invariant under the action of \mathbb{C}^\times . It does not intersect the coordinate lines \mathbb{C}_1 and \mathbb{C}_2 , except at the point $\mathbf{0}$, and so it does not contain singular orbits of h_t . The intersection

$$K = V(f) \cap \mathbb{S}^3$$

is a regular orbit of h_t , and hence a the (p, q) -torus knot in \mathbb{S}^3 .

To simplify the notation, we take $c_1 = c_2 = 1$. The K can be expressed as

$$K = \begin{cases} z_1 = a_1 e^{2\pi i p t} \\ z_2 = a_2 e^{2\pi i q t + \frac{i\pi}{p}} \end{cases} \quad | \quad \begin{array}{l} a_1^2 + a_2^2 = 1 \\ a_1^q - a_2^p = 0 \end{array}, \quad a_1, a_2 > 0 \quad (9)$$

lying on the torus $T := \{|z_1| = a_1, |z_2| = a_2\}$. Since K is itself one regular orbit of h_t , its complement $\mathbb{S}^3 \setminus K$ is h_t -invariant. As a direct consequence of proposition 2.4 we obtain

Proposition 2.5. *The flow h_t defines a Seifert fibration in $\mathbb{S}^3 \setminus K$. The base orbifold, Θ , is a 2-dimensional orbifold with underlying space \mathbb{R}^2 (a punctured 2-sphere) and two singular points: cone points of indexes p and q .*

We have a decomposition

$$\mathbb{S}^3 \setminus K = \mathbb{T}(p, q) \sqcup (T \setminus K) \sqcup \mathbb{T}(q, p),$$

where

$$\begin{aligned} S_1 &\hookrightarrow \mathbb{T}(p, q) := \{ |z_1| < a_1, |z_2| = \sqrt{1 - |z_1|^2} \} , \\ K &\hookrightarrow T := \{ |z_1| = a_1, |z_2| = a_2 \} , \\ S_2 &\hookrightarrow \mathbb{T}(q, p) := \{ |z_1| = \sqrt{1 - |z_2|^2}, |z_2| < \mu \} . \end{aligned} \tag{10}$$

From this decomposition of $\mathbb{S}^3 \setminus K$ we can produce the well-known presentation of the torus knot group:

$$G_{p,q} := G_K = \pi_1(\mathbb{S}^3 \setminus K_{p,q}) \cong \langle S_1, S_2 ; S_1^p = S_2^q \rangle . \tag{11}$$

The element $S = S_1^p = S_2^q$ represents a regular fibre. This element is a generator of $\pi_1(T \setminus K)$ and of the center $Z(G_K)$ (see remark 2.1 and lemma 2.3). We have the central extension

$$1 \longrightarrow \langle S \rangle \longrightarrow G_K \longrightarrow \pi_1(\Theta) \longrightarrow 1 .$$

Thus $\pi_1(\Theta) \cong G_K / Z(G_K) \cong \mathbb{Z}_p * \mathbb{Z}_q$. The universal orbifold-cover of Θ will be constructed in section 4.1, see proposition 4.1.

3 Automorphic forms on $\widetilde{SL}_2(\mathbb{R})$

The Lie group $\widetilde{SL}_2(\mathbb{R})$ is defined as the universal covering group of the simple Lie group $PSL_2(\mathbb{R})$. In this section we describe the group $\widetilde{SL}_2(\mathbb{R})$ by identifying it with the universal cover of the unit tangent bundle $U\mathbb{H}^2$ on the hyperbolic plane. We start by identifying $PSL_2(\mathbb{R})$ with $U\mathbb{H}^2$. Thus $\widetilde{SL}_2(\mathbb{R})$ is endowed with the structure of a topologically trivial \mathbb{R} -fibration over the hyperbolic plane. This construction can be used to obtain a left invariant metric on $\widetilde{SL}_2(\mathbb{R})$ which makes it a 3-dimensional geometry, one of the eight geometries of Thurston's list. Furthermore, this allows us to view automorphic forms for discrete groups acting on the upper half plane as functions on $\widetilde{SL}_2(\mathbb{R})$. The fact that we pass to the universal cover allows the consideration of forms of fractional degrees which we need for our main construction.

3.1 Isometric group actions on the hyperbolic plane

We work with the upper half plane model of the hyperbolic plane

$$\mathbb{H}^2 = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} , \quad ds^2 = \frac{|dz|^2}{\text{Im}(z)^2} .$$

The matrix Lie group

$$SL_2(\mathbb{R}) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

acts isometrically on \mathbb{H}^2 by Möbius transformations:

$$A.z := \frac{az + b}{cz + d}.$$

This action is transitive. The stabilizer of the point $i \in \mathbb{H}^2$ is $SO_2(\mathbb{R})$, isomorphic to the circle group \mathbb{S}^1 . The center of $SL_2(\mathbb{R})$ consists of the elements $I, -I$, and acts trivially on \mathbb{H}^2 . To make the action effective, we take the quotient

$$P : SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R}) \cong SL_2(\mathbb{R})/\{\pm I\}.$$

The action of $PSL_2(\mathbb{R})$ is effective and transitive. The stabilizer of the point i is $PSO_2(\mathbb{R})$, also isomorphic to \mathbb{S}^1 . The coset space $PSL_2(\mathbb{R})/PSO_2(\mathbb{R})$ is diffeomorphic to the hyperbolic plane, the stabilizer being simply transitive on the circle $U_i\mathbb{H}^2$ of unit tangent vectors at the point i . Thus $PSL_2(\mathbb{R})$ is diffeomorphic to the unit tangent bundle on \mathbb{H}^2 :

$$U\mathbb{H}^2 := \{(z, \vec{v}) : z \in \mathbb{H}^2, \vec{v} \in T_z\mathbb{H}^2, \|\vec{v}\|_{\mathbb{H}} = 1\}.$$

To fix a diffeomorphism we choose $(i, \frac{\partial}{\partial z}|_i) \in U\mathbb{H}^2$ to be a base point, and identify $PSL_2(\mathbb{R})$ with its orbit through that point. Since the hyperbolic plane is homeomorphic to a 2-cell, the fibre bundles $T\mathbb{H}^2$ and $U\mathbb{H}^2$ are topologically trivial. Hence $U\mathbb{H}^2$ is diffeomorphic to $\mathbb{H}^2 \times \mathbb{S}^1$ and the fundamental group $\pi_1(U\mathbb{H}^2) \cong \pi_1(PSL_2(\mathbb{R}))$ is an infinite cyclic group.

The Lie group $\widetilde{SL}_2(\mathbb{R})$ is defined as the universal covering group of $PSL_2(\mathbb{R})$. We denote the canonical projection by $\tilde{P} : \widetilde{SL}_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$. The fundamental group of $PSL_2(\mathbb{R})$ is identified with the center C of $\widetilde{SL}_2(\mathbb{R})$. Thus we have a central extension

$$1 \longrightarrow C \longrightarrow \widetilde{SL}_2(\mathbb{R}) \xrightarrow{\tilde{P}} PSL_2(\mathbb{R}) \longrightarrow 1. \quad (12)$$

The group $\widetilde{SL}_2(\mathbb{R})$ acts, via \tilde{P} , on \mathbb{H}^2 and $U\mathbb{H}^2$. Since $PSL_2(\mathbb{R})$ is identified with $U\mathbb{H}^2$, it follows that $\widetilde{SL}_2(\mathbb{R})$ is identified with the universal covering $\widetilde{U\mathbb{H}^2}$ (which is in turn diffeomorphic to $\mathbb{H}^2 \times \mathbb{R}$). This endows $\widetilde{SL}_2(\mathbb{R})$ with the structure of a topologically trivial \mathbb{R} -bundle over the hyperbolic plane. We fix the generator c of C corresponding to a simultaneous counter-clockwise rotation of all unit tangent vectors to \mathbb{H}^2 at angle 2π keeping the base points fixed.

For any integer $r \geq 1$ we can consider the group $\widetilde{SL}_2(\mathbb{R})/\langle c^r \rangle$, which is a r -fold covering of $PSL_2(\mathbb{R})$. Clearly $\widetilde{SL}_2(\mathbb{R})/\langle c^r \rangle$ has cyclic center of order r and is diffeomorphic to $\mathbb{H}^2 \times \mathbb{S}^1$.

Remark 3.1. *The hyperbolic metric on \mathbb{H}^2 can be used to induce a left invariant metric on $PSL_2(\mathbb{R})$ and from there on $SL_2(\mathbb{R})$. This turns $SL_2(\mathbb{R})$ into a model for a geometry, in the sense discussed in section 2.1. The exact form of this metric is not important for the purposes of this paper, so we omit the precise formulation.*

3.2 Automorphic forms for discrete subgroups of $\widetilde{SL}_2(\mathbb{R})$

If G is any subgroup of $\widetilde{SL}_2(\mathbb{R})$, then $G \cap C \subset Z(G)$ and $\widetilde{P}(G) \cong G/(G \cap C)$. Clearly there are discrete subgroups of $\widetilde{SL}_2(\mathbb{R})$ which project to non-discrete subgroups of $PSL_2(\mathbb{R})$. On the other hand, if Γ is a discrete subgroup of $PSL_2(\mathbb{R})$, i.e. a Fuchsian group, then its preimage $\widetilde{\Gamma} = \widetilde{P}^{-1}(\Gamma)$ is discrete in $\widetilde{SL}_2(\mathbb{R})$. We are mainly interested in discrete subgroups $\widetilde{\Gamma}$ of $\widetilde{SL}_2(\mathbb{R})$ arising as preimages of Fuchsian groups and, more generally, subgroups $G \subset \widetilde{\Gamma}$ which are obviously discrete as well, and project to Fuchsian groups.

Let G be a discrete subgroup of $\widetilde{SL}_2(\mathbb{R})$. We assume that the projection $\widetilde{P}(G) \subset PSL_2(\mathbb{R})$ is a Fuchsian group of the first kind, i.e. has a fundamental domain with finite volume. In such case the quotient $\mathbb{H}^2/\widetilde{P}(G)$ can be compactified (holomorphically) by adding a finite number of points. This section is devoted to the notion of an automorphic form for G . We consider forms of rational degrees as defined by Milnor in [9]; this approach is somewhat non-standard, but suitable for our purposes.

Let $\widetilde{\mathbb{C}}^\times$ denote the universal covering group of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. As a complex Lie group $\widetilde{\mathbb{C}}^\times$ is isomorphic to the additive group \mathbb{C} , but we prefer to keep the multiplicative notation. Recall that for $\zeta \in \widetilde{\mathbb{C}}^\times$ we have a well-defined r -th root $\zeta^{\frac{1}{r}}$ for any positive integer r , and hence we have a well-defined k -th power ζ^k for any rational number k .

Let $T'\mathbb{H}^2$ denote the tangent bundle on the hyperbolic plane with removed zero-section. This bundle has a nonvanishing holomorphic section $\frac{\partial}{\partial z}$. Hence $T'\mathbb{H}^2$ is holomorphically trivial, i.e. biholomorphic to $\mathbb{H}^2 \times \mathbb{C}^\times$. Notice that $T'\mathbb{H}^2$ contains the unit tangent bundle $U\mathbb{H}^2$ considered earlier. The isomorphism $T'\mathbb{H}^2 \cong \mathbb{H}^2 \times \mathbb{C}^\times$ associated with the section $\frac{\partial}{\partial z}$ is such that $U_z\mathbb{H}^2$ is identified with $\{z\} \times \{\zeta \in \mathbb{C}^\times : |\zeta| = \text{Im}(z)\}$. The universal cover $\widetilde{T'\mathbb{H}^2}$ is biholomorphic to $\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$. The group $\widetilde{SL}_2(\mathbb{R})$ acts on $\widetilde{T'\mathbb{H}^2}$, and hence on $\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$, in an obvious way and all stabilizers are trivial. Thus any orbit of this action is diffeomorphic to $SL_2(\mathbb{R})$. We fix the embedding $\widetilde{SL}_2(\mathbb{R}) \subset \mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$ given by the orbit map through the point $(i, 1) \in \mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$.

Let $\gamma \in \widetilde{SL}_2(\mathbb{R})$. The expression for the transformation of $\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$ associated with γ is

$$\gamma(z, w) = (\gamma(z), \gamma'(z)w),$$

where $\gamma(z)$ is given by the usual action of $\widetilde{SL}_2(\mathbb{R})$ on \mathbb{H}^2 and $\gamma' : \mathbb{H}^2 \rightarrow \widetilde{\mathbb{C}}^\times$ is the lift of the (nonvanishing) derivative $\frac{d\gamma(z)}{dz}$ determined by the requirement to satisfy the chain rule $(\gamma_2 \gamma_1)'(z) = \gamma_2'(\gamma_1(z))\gamma_1'(z)$.

Definition 3.1. A (differential) form of degree $k \in \mathbb{Q}$, or a k -form, on \mathbb{H}^2 is defined to be a complex valued function ω of two variables $z \in \mathbb{H}^2$ and $dz \in \widetilde{\mathbb{C}}^\times$ of the form

$$\omega(z, dz) = f(z)dz^k,$$

where f is a holomorphic function on \mathbb{H}^2 . The product of $f(z)$ and dz^k is taken after projecting dz^k from $\widetilde{\mathbb{C}}^\times$ to \mathbb{C}^\times via the universal covering map.

Let $\omega = f(z)dz^k$ be a form on \mathbb{H}^2 . For any $\gamma \in \widetilde{SL}_2(\mathbb{R})$ we have the pullback

$$\gamma^*\omega(z, dz) = \omega(\gamma.(z, dz)) = f(\gamma(z))(\gamma'(z))^k dz^k.$$

Definition 3.2. Let $\omega(z, dz) = f(z)dz^k$ be a form on \mathbb{H}^2 . If $\chi \in \text{Hom}(G, U(1))$ is a character of G , a form ω on \mathbb{H}^2 is called χ -automorphic, if the following two conditions are satisfied:

(i) $\gamma^*\omega = \chi(\gamma)\omega$ for all $\gamma \in G$.

(ii) Suppose $P(G)$ has a fundamental domain with a cusp at ∞ (the case of a different cusp is treated by sending it to ∞). Let $z \mapsto z + \lambda$ be a generator of the parabolic subgroup of G fixing ∞ . Then the function $f(z)$ is required to have an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \frac{z}{\lambda}}$$

which is convergent for $z \in \mathbb{H}^2$. In such case f is said to be holomorphic at ∞ .

If χ is the trivial character of G , an χ -automorphic form will be called **G -automorphic**; in such case we have $\gamma^*\omega = \omega$ for all $\gamma \in G$. When the group is understood from the context, by automorphic form we will mean a form which is χ -automorphic for some character χ . An automorphic form vanishing at a cusp of $\widetilde{P}(G)$ is called a **cusp form**.

The automorphy property of a form $\omega(z, dz) = f(z)dz^k$ is reflected on a property of the function f . Namely,

$$\gamma^*\omega = \chi(\gamma)\omega \iff f(\gamma(z))(\gamma'(z))^k = \chi(\gamma)f(z).$$

The space of χ -automorphic k -forms will be denoted by $\mathcal{A}_G^{k,\chi}$. The algebra of forms on \mathbb{H}^2 generated by all automorphic forms for G will be denoted by $\mathcal{A}_G^{*,*}$. This is a bi-graded algebra with components $\mathcal{A}_G^{k,\chi}$. The G -automorphic forms generate a subalgebra, to be denoted by \mathcal{A}_G^* , with degree components $\mathcal{A}_G^k = \mathcal{A}_G^{k,1}$. Since we have embedded $\widetilde{SL}_2(\mathbb{R})$ into $\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$, any form ω on \mathbb{H}^2 can be viewed, via restriction, as a complex valued function on $\widetilde{SL}_2(\mathbb{R})$. The elements of \mathcal{A}_G^* are then well-defined functions on the coset space $\widetilde{SL}_2(\mathbb{R})/G$.

Lemma 3.1. Let r, s be co-prime integers, $r > 0$, and let $\omega(z, dz) = f(z)dz^{\frac{s}{r}}$ be a form of degree $\frac{s}{r}$ on \mathbb{H}^2 .

(i) The form ω is invariant under the action of the central subgroup $\langle c^r \rangle \subset \widetilde{SL}_2(\mathbb{R})$, and hence is a well-defined function on the group $\widetilde{SL}_2(\mathbb{R})/\langle c^r \rangle$.

(ii) If ω is G -automorphic, then $G \cap C \subset \langle c^r \rangle$.

Proof. The generator c of the center of $\widetilde{SL}_2(\mathbb{R})$ satisfies $c(z) = z$ and $c'(z)^k$ projects to $e^{2\pi i k} \in \mathbb{C}^\times$ for all $z \in \mathbb{H}^2$ and $k \in \mathbb{Q}$. Hence, for $n \in \mathbb{N}$, we have

$$(c^n)^*\omega(z, dz) = f(c^n(z))c'(z)^n dz^{\frac{s}{r}} = f(z)e^{2\pi i \frac{ns}{r}} dz^{\frac{s}{r}} = e^{2\pi i \frac{ns}{r}} \omega(z, dz).$$

Now, since s and r are assumed co-prime, we see that $(c^n)^*\omega = \omega$ if and only if r divides n . Both parts of the lemma follow immediately. \square

The following lemma is almost literally taken from [9]; it describes roots of automorphic forms.

Lemma 3.2. *Let $\omega(z, dz) = f(z)dz^k$ be a χ -automorphic form. Suppose that f possesses an n -root, i.e. there is a holomorphic function f_1 on \mathbb{H}^2 and such that $f_1(z)^n = f(z)$. Then the form $\omega_1(z, dz) = f_1(z)dz^{\frac{k}{n}}$ is χ_1 -automorphic for some character χ_1 of G satisfying $\chi_1^n = \chi$.*

Proof. Let $\gamma \in G$. Then ω_1 and $\gamma^*\omega_1$ are both forms of degree $\frac{k}{n}$ on \mathbb{H}^2 . Hence the quotient $\frac{\gamma^*\omega_1}{\omega_1}$ is a well-defined meromorphic function on \mathbb{H}^2 . The n -th power of this function is $\frac{\gamma^*\omega}{\omega} = \chi(\gamma)$ which is constant. Therefore $\frac{\gamma^*\omega_1}{\omega_1}$ is constant as well. Define $\chi_1(\gamma) = \frac{\gamma^*\omega_1}{\omega_1}$. Then χ_1 is a character of G and we have $\chi_1^n = \chi$. \square

4 Triangle groups and torus knot groups

4.1 Triangle groups

In this section, we discuss a beautiful class of Fuchsian groups - the triangle groups. They have been extensively studied, a basic introduction may be found in Berdon's book [1].

For any two distinct points a and b in \mathbb{H}^2 denote by \overline{ab} the directed geodesic segment connecting a and b . The same notation will be used for the limit case, when one or both points lie on the absolute. Consider a geodesic triangle Δ in \mathbb{H}^2 , with vertices a, b, c (some of them may lie on the absolute). To fix the notation, suppose that the triple a, b, c is positively oriented. Suppose that the interior angles of Δ , at the vertices a, b, c respectively, are equal to $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$, with $p, q, r \in \mathbb{N} \cup \{\infty\}$ ¹. Denote by $\sigma_a, \sigma_b, \sigma_c$ the reflections in \mathbb{H}^2 with respect to $\overline{bc}, \overline{ac}, \overline{ab}$. The triangle Δ is a fundamental polygon for a discrete group of isometries of \mathbb{H}^2 , with presentation

$$\Gamma^*(p, q, r) \cong \langle \sigma_a, \sigma_b, \sigma_c ; \sigma_a^2 = \sigma_b^2 = \sigma_c^2 = (\sigma_b \sigma_c)^p = (\sigma_c \sigma_a)^q = (\sigma_a \sigma_b)^r = 1 \rangle$$

Such a group is called a (p, q, r) -triangle group. The elements $\alpha_0 := \sigma_b \sigma_c$, $\beta_0 := \sigma_c \sigma_a$, $\gamma_0 := \sigma_a \sigma_b$ are elliptic or parabolic² elements of $PSL_2(\mathbb{R})$, with fixed points a, b, c and angles of rotation $\frac{2\pi}{p}, \frac{2\pi}{q}, \frac{2\pi}{r}$ respectively. We have $\gamma_0 = \beta_0^{-1} \alpha_0^{-1}$.

If Δ_1 is another triangle in \mathbb{H}^2 , similar (same angles) to Δ , then there exists an isometry of \mathbb{H}^2 sending Δ to Δ_1 . Thus any two (p, q, r) -triangle groups are conjugate in $\text{Isom}(\mathbb{H}^2)$. The generators of $\Gamma^*(p, q, r)$ do not preserve the orientation in \mathbb{H}^2 , so this group is not Fuchsian.

To obtain a Fuchsian group we take the subgroup of index 2 of $\Gamma^*(p, q, r)$ consisting of all orientation preserving elements. We denote this group by $\Gamma(p, q, r)$ and, following the

¹We assume that $\frac{\pi}{\infty} = 0$ for a vertex on the absolute. To obtain a hyperbolic triangle, one must have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. This is certainly true when $1 < p < q, r = \infty$.

²In the case when some of the parameters p, q, r equal ∞ , for instance $r = \infty$, the corresponding element $\gamma_0 = \sigma_a \sigma_b$ is parabolic. The relation $\gamma_0^r = 1$ is omitted.

tradition, also refer to it as a (p, q, r) -triangle group; the distinction should be easily made from the context. We are mostly concerned with the orientation preserving triangle groups. To obtain a fundamental domain for $\Gamma(p, q, r)$, denote $\Delta' := \sigma_c(\Delta)$, $d := \sigma_c(c)$, and set

$$D := \Delta \cup \Delta'.$$

Then D is a quadrangle in \mathbb{H}^2 with vertices a, d, b, c and angles $\frac{2\pi}{p}, \frac{\pi}{r}, \frac{2\pi}{q}, \frac{\pi}{r}$ respectively. The edges of D are identified by α_0 and β_0 as follows

$$\begin{aligned} \alpha_0 : \overline{ad} &\longmapsto \overline{ac} \\ \beta_0 : \overline{bc} &\longmapsto \overline{bd}. \end{aligned} \quad (13)$$

According to Poincaré's theorem for fundamental polygons, α_0 and β_0 generate a Fuchsian group with fundamental domain D and presentation

$$\Gamma(p, q, r) \cong \langle \alpha_0, \beta_0 ; \alpha_0^p = 1, \beta_0^q = 1, (\alpha_0\beta_0)^r = 1 \rangle. \quad (14)$$

It is easy to see that $\sigma_b(\Delta) = \alpha_0(\Delta')$ and $\sigma_a(\Delta) = \beta_0^{-1}(\Delta')$, thus $D_1 := \Delta \cup \sigma_b(\Delta)$ and $D_2 := \Delta \cup \sigma_a(\Delta)$ are alternative fundamental domains for $\Gamma(p, q, r)$. The independence on the choice of reflection generating the factorgroup is obvious from the definition, and also in view of the presentation

$$\Gamma(p, q, r) \cong \langle \alpha_0, \beta_0, \gamma_0 ; \alpha_0^p = 1, \beta_0^q = 1, \gamma_0^r = 1, \alpha_0\beta_0\gamma_0 = 1 \rangle. \quad (15)$$

A classical example is the modular group $PSL(2, \mathbb{Z})$. Its traditional fundamental domain and generators are

$$D_1(PSL(2, \mathbb{Z})) = \left\{ x + iy \in \mathbb{H}^2 : -\frac{1}{2} < x < \frac{1}{2}, y > \sqrt{1 - x^2} \right\},$$

$$P(S), P(T) \text{ , where } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The relations are $S^2 = 1$ and $(ST^{-1})^3 = 1$. This group is Fuchsian, it preserves the orientation of \mathbb{H}^2 . In the notation introduced above, $PSL(2, \mathbb{Z}) \cong \Gamma(2, 3, \infty)$ is a triangle group. It is a subgroup of index 2 of the triangle group $\Gamma(2, 3, \infty)$ with fundamental domain

$$\Delta(PSL(2, \mathbb{Z})) = \left\{ x + iy \in \mathbb{H}^2 : 0 < x < \frac{1}{2}, y > \sqrt{1 - x^2} \right\}.$$

Alternative fundamental domain and generators for $PSL(2, \mathbb{Z})$ are

$$D(PSL(2, \mathbb{Z})) = \left\{ x + iy \in \mathbb{H}^2 : 0 < x < \frac{1}{2}, y > \sqrt{1 - (x - 1)^2} \right\},$$

$$P(S), P(U) \text{ , where } U = ST^{-1}.$$

Hence, if we denote $\alpha_0 := P(S)$, $\beta_0 := P(U)$, we have

$$PSL(2, \mathbb{Z}) \cong \langle \alpha_0, \beta_0 ; \alpha_0^2 = 1, \beta_0^3 = 1 \rangle \cong \Gamma(2, 3, \infty) .$$

We now focus on the class triangle groups which actually concern us for the rest of the paper. Let p and q be co-prime numbers, and set

$$\Gamma_{p,q} := \Gamma(p, q, \infty) \cong \langle \alpha_0, \beta_0 ; \alpha_0^p = 1, \beta_0^q = 1 \rangle . \quad (16)$$

Abstractly, $\Gamma_{p,q}$ is isomorphic to the free product $\mathbb{Z}_p * \mathbb{Z}_q$. The triangle Δ has exactly one vertex, say c , lying on the absolute. The fundamental domain D corresponding to the above presentation is a quadrangle with two cusps c and d , and two vertices a and b inside \mathbb{H}^2 with angles $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$ respectively. We can take

$$a = -\cos \frac{\pi}{p} + i \sin \frac{\pi}{p}, \quad d = 0, \quad b = \cos \frac{\pi}{q} + i \sin \frac{\pi}{q}, \quad c = \infty$$

$$D = \left\{ x + iy \in \mathbb{H}^2 : \begin{array}{l} -\cos \frac{\pi}{p} < x < \cos \frac{\pi}{q} \\ y > \sqrt{\left(\frac{1}{2 \cos \frac{\pi}{p}}\right)^2 - \left(x + \frac{1}{2 \cos \frac{\pi}{p}}\right)^2} \\ y > \sqrt{\left(\frac{1}{2 \cos \frac{\pi}{q}}\right)^2 - \left(x - \frac{1}{2 \cos \frac{\pi}{q}}\right)^2} \end{array} \right\} , \quad (17)$$

$$\alpha_0 = P(A_0), \quad \beta_0 = P(B_0), \quad A_0 = \begin{pmatrix} 2 \cos \frac{\pi}{p} & 1 \\ -1 & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \cos \frac{\pi}{q} \end{pmatrix} .$$

The action of the generators is given in (13). The two cusps are identified by the generators.

Remark 4.1. *An alternative fundamental domain with one cusp for $\Gamma_{p,q}$ is*

$$D_1 := \Delta \cup \sigma_b(\Delta) = (D \cap \alpha_0(D)) \cup \alpha_0(D)$$

It corresponds to the following generators and relations

$$\Gamma_{p,q} \cong \langle \alpha_0, \gamma_0 ; \alpha_0^p = 1, (\gamma_0 \alpha_0)^q = 1 \rangle .$$

The element $\gamma_0 = (\alpha_0 \beta_0)^{-1}$ is parabolic with fixed point ∞ .

The quotient $\mathbb{H}^2 / \Gamma_{p,q}$ is an orbifold with underlying topological space $\mathbb{S}^2 \setminus \{\text{point}\}$. The missing point corresponds to the cusp of the fundamental domain D_1 . This orbifold is an important ingredient in our construction, it was already encountered as the base orbifold Θ of the Seifert fibration in a torus knot complement, see proposition 2.5.

Proposition 4.1. *The projection $\mathbb{H}^2 \rightarrow \mathbb{H}^2 / \Gamma_{p,q}$ is the universal covering of the orbifold Θ , i.e. $\Theta \cong \mathbb{H}^2 / \Gamma_{p,q}$.*

Proof. The generators of $\Gamma_{p,q}$ give the following identifications of edges of D :

$$\alpha_0 : \overline{ad} \longmapsto \overline{ac} \quad , \quad \beta_0 : \overline{bc} \longmapsto \overline{bd} \quad .$$

The diagonal \overline{cd} cuts D in two triangles, say D_a and D_b , containing the vertices a and b respectively. These triangles are mapped under the action of α_0 and β_0 on two cones D^2/\mathbb{Z}_p and D^2/\mathbb{Z}_q . These two cones are the base orbifolds of the fibred tori $\mathbb{T}(q,p)$ and $\mathbb{T}(p,q)$ given in (10). The projection of the annulus $T \setminus K$, from formula (10), is the diagonal \overline{cd} . The cusps c and d are identified by both α_0 and β_0 , and correspond to the knot K . \square

4.2 The commutator subgroup of a (p,q,∞) -triangle group

Let $\Gamma'_{p,q}$ denote the commutator subgroup of $\Gamma_{p,q}$. The main result in this section states that $\Gamma'_{p,q}$ is a free group of rank $(p-1)(q-1)$. This result is somewhat tangential to the main line of the paper, and in fact can be deduced easily from well known properties of torus knot groups, cf [4]. However, the proof we give here is somewhat interesting in that it uses the geometry of triangle groups. In particular we construct fundamental domain for the action of $\Gamma'_{p,q}$ on \mathbb{H}^2 and give free generators. The freeness of the commutator subgroup of a torus knot group can be deduced from this result.

Since $\Gamma_{p,q}$ is isomorphic to the free product $\mathbb{Z}_p * \mathbb{Z}_q$, it follows that the factorgroup $\Gamma_{p,q}/\Gamma'_{p,q}$ is isomorphic to the direct product $\mathbb{Z}_p \times \mathbb{Z}_q$. Since p and q are co-prime, we can conclude that $\Gamma_{p,q}/\Gamma'_{p,q}$ is cyclic of order pq . For $\gamma \in \Gamma_{p,q}$ let $\overline{\gamma} \in \Gamma_{p,q}/\Gamma'_{p,q}$ denote the image of the canonical projection.

Remark 4.2. (i) *If Γ_1 is a normal subgroup of $\Gamma_{p,q}$ such that $\Gamma_{p,q}/\Gamma_1$ is cyclic of order pq , then Γ_1 coincides with the commutator $\Gamma'_{p,q}$.*

(ii) *The factorgroup $\Gamma_{p,q}/\Gamma'_{p,q}$ consists of the elements $\overline{\alpha}_0^i \overline{\beta}_0^j$, $i = 0, \dots, p-1$, $j = 0, \dots, q-1$.*

(iii) *Each of the two elements below generates $\Gamma_{p,q}/\Gamma'_{p,q}$*

$$\overline{\gamma}_0 = (\overline{\alpha}_0 \overline{\beta}_0)^{-1} \quad , \quad \overline{x}_0 = \overline{\alpha}_0^{q_1} \overline{\beta}_0^{p_1} \quad ,$$

where p_1, q_1 are integers such that $pp_1 + qq_1 = 1$. We have $\overline{\gamma}_0^{-pp_1^2 - qq_1^2} = \overline{x}_0$ and $\overline{x}_0^{pq-p-q} = \overline{\gamma}_0$.

Proposition 4.2. *The commutator subgroup $\Gamma'_{p,q}$ is free of rank $(p-1)(q-1)$. Any of the following two sets generates $\Gamma'_{p,q}$ freely.*

$$\xi_{0(i,j)} := [\alpha_0^i, \beta_0^j] = \alpha_0^i \beta_0^j \alpha_0^{-i} \beta_0^{-j} \quad , \quad i = 1, \dots, p-1 \quad , \quad j = 1, \dots, q-1 \quad , \quad (18)$$

$$\eta_{0(k,j)} := [\alpha_0, \alpha_0^k \beta_0^j \alpha_0^{-k}] = \alpha_0^k [\alpha_0, \beta_0^j] \alpha_0^{-k} \quad , \quad k = 0, \dots, p-2 \quad , \quad j = 1, \dots, q-1 \quad . \quad (19)$$

Proof. The commutator $\Gamma'_{p,q}$ is the smallest normal subgroup of $\Gamma_{p,q}$ containing $[\alpha_0, \beta_0]$. Denote by Ξ the subgroup of $\Gamma_{p,q}$ generated by the elements $\xi_{0(i,j)}$. The inclusion $\Xi \subset \Gamma'_{p,q}$ is clear. To prove that $\Gamma'_{p,q} \subset \Xi$, we will show that Ξ is a normal subgroup of $\Gamma_{p,q}$. It is sufficient to see that the conjugates of $\xi_{0(i,j)}$ by α_0 and β_0 are still in Ξ . Direct computations show that

$$\alpha_0 \xi_{0(i,j)} \alpha_0^{-1} = \xi_{0(i+1,j)} \xi_{0(1,j)}^{-1} \quad , \quad \beta_0 \xi_{0(i,j)} \beta_0^{-1} = \xi_{0(i,1)}^{-1} \xi_{0(i,j+1)} \quad (20)$$

Hence, Ξ is a normal in $\Gamma_{p,q}$ and $\Xi \equiv \Gamma'_{p,q}$.

The elements (18) are expressed in terms of (19) in the following way. First we have

$$\eta_{0(0,j)} = \xi_{0(1,j)} \quad , \quad j = 1, \dots, q-1 \quad .$$

Using (20) and induction on i , it is easy to show that

$$\eta_{0(i-1,j)} \eta_{0(i-2,j)} \dots \eta_{0(1,j)} \eta_{0(0,j)} = \xi_{0(i,j)} \quad , \quad i = 1, \dots, p-1, \quad j = 1, \dots, q-1 \quad .$$

Hence, the elements $\eta_{0(i,j)}$ also generate $\Gamma'_{p,q}$.

To obtain a fundamental domain for $\Gamma'_{p,q}$ one may take the union of the images of the fundamental domain D of $\Gamma_{p,q}$ under the action of the elements representing the factorgroup $\Gamma_{p,q}/\Gamma'_{p,q}$. We denote

$$D' := \bigcup_{i,j} D_{i,j} \quad , \quad \text{where } D_{i,j} := \alpha_0^i \beta_0^j(D) \quad , \quad i = 0, \dots, p-1, \quad j = 0, \dots, q-1 \quad . \quad (21)$$

D' is a polygon in \mathbb{H}^2 with $2p(q-1)$ vertices, of which $p(q-1)$ cusps and $p(q-1)$ points inside \mathbb{H}^2 , changing alternatively. The interior angles at the vertices inside \mathbb{H}^2 are all equal to $\frac{2\pi}{p}$. The edges of D' are

$$\begin{aligned} l'_{i,j} &:= \alpha_0^i \beta_0^j(\overline{ac}) \quad , \quad i = 0, \dots, p-1 \quad , \\ l''_{i,j} &:= \alpha_0^i \beta_0^j(\overline{ad}) \quad , \quad j = 1, \dots, q-1 \quad . \end{aligned}$$

The arcs $\alpha_0^i(\overline{ad})$ and $\alpha_0^i(\overline{ac})$ lie in the interior of D' . A direct computation shows that

$$\eta_{0(i,j)} : l'_{i,j} \longmapsto l''_{i+1,j} \quad , \quad i \in \mathbb{Z}(\text{mod } p) \quad , \quad j = 1, \dots, q-1 \quad .$$

According to Poincaré's theorem, D' is a fundamental polygon for a Fuchsian group with generators

$$\eta_{0(i,j)} \quad , \quad i = 0, \dots, p-1, \quad j = 1, \dots, q-1 \quad .$$

and relations

$$\eta_{0(p-1,j)} \eta_{0(p-2,j)} \dots \eta_{0(1,j)} \eta_{0(0,j)} = 1 \quad , \quad j = 1, \dots, q-1 \quad .$$

These relations correspond to the cycles of vertices of D' (inside \mathbb{H}^2). At every cycle of vertices the sum of the angles is equal to $p\frac{2\pi}{p} = 2\pi$. Hence, the group acts freely on \mathbb{H}^2 and the quotient is a smooth surface. Moreover, we can express $\eta_{0(p-1,j)}$ from the relations, and we are left with a free group of rank $(p-1)(q-1)$ generated by the elements

$$\eta_{0(i,j)} \quad , \quad i = 0, \dots, p-2, \quad j = 1, \dots, q-1 .$$

This completes the proof. \square

Remark 4.3. *Proposition 4.2 implies that the quotient $F := \mathbb{H}^2/\Gamma'_{p,q}$ is an orientable smooth open surface, whose homotopy type is a bouquet of $(p-1)(q-1)$ circles. In fact, F can be embedded in \mathbb{S}^3 as a Seifert surface for a (p,q) -torus knot.*

Since p and q are co-prime, all the cusps of the fundamental polygon D' are equivalent under the action of $\Gamma'_{p,q}$. The corresponding product of the elements $\eta_{0(i,j)}$ is too complicated. To obtain simple formulae we shall construct another fundamental domain D'_1 for $\Gamma'_{p,q}$ with only one cusp. We use the fundamental domain D_1 for $\Gamma_{p,q}$ given in remark 4.1. In remark 4.2 we observed that the image of the element $\gamma_0 = (\alpha_0\beta_0)^{-1}$ generates $\Gamma_{p,q}/\Gamma'_{p,q} \cong \mathbb{Z}_{pq}$. Thus

$$D'_1 := \bigcup_{k=0}^{pq-1} \gamma_0^k(D_1)$$

is a fundamental polygon for $\Gamma'_{p,q}$. The only cusp of D_1 , the point ∞ , is the fixed point of the element γ_0 . Hence the polygon D'_1 has only one cusp, ∞ . The two infinite edges of D'_1 are identified by the element $\gamma_0^{pq} = (\alpha_0\beta_0)^{-pq}$ which belongs to $\Gamma'_{p,q}$.

4.3 A representation of the torus knot group in $\widetilde{SL}_2(\mathbb{R})$

The following lemma gives a representation of the torus knot group $G_K := \pi_1(\mathbb{S}^3 \setminus K)$ as a discrete subgroup of $\widetilde{SL}_2(\mathbb{R})$.

Lemma 4.3. *Let $\Gamma_{p,q}$ be the triangle Fuchsian group described in (16). The pre-image $\widetilde{\Gamma} := \widetilde{P}^{-1}(\Gamma_{p,q})$ in $\widetilde{SL}_2(\mathbb{R})$ is a discrete subgroup, isomorphic to the torus knot group presented in (11). There is a commutative diagram*

$$\begin{array}{ccccccc} 1 & \longrightarrow & Z(\widetilde{\Gamma}) & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \Gamma_{p,q} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & C & \longrightarrow & \widetilde{SL}_2(\mathbb{R}) & \longrightarrow & PSL_2(\mathbb{R}) & \longrightarrow & 1 \end{array}$$

where the rows are central extensions, the leftmost downarrow is an isomorphism and the other two - monomorphisms.

Proof. The group $\Gamma_{p,q}$ is generated by the elliptic elements α_0 and β_0 . The angles of rotation are $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$. The elements α_0^p and β_0^q represent rotations at angle 2π centered at the respective fixed points. Hence, these elements correspond to the generator c of $C = \pi_1(PSL_2(\mathbb{R}))$. We have already identified c as an element of $\widetilde{SL}_2(\mathbb{R})$. Let α and β be the unique elements in $\widetilde{P}^{-1}(\alpha_0)$ and $\widetilde{P}^{-1}(\beta_0)$ respectively, such that $\alpha^p = \beta^q = c$. It is clear that α and β generate $\widetilde{\Gamma}$. The relation $\alpha^p = \beta^q$ is (essentially) the only relation in $\widetilde{\Gamma}$, because the factorization by C maps $\widetilde{SL}_2(\mathbb{R})$ on $PSL_2(\mathbb{R})$ and $\widetilde{\Gamma}$ on $\Gamma_{p,q}$. Obviously $\widetilde{\Gamma}$ is a discrete subgroup of $\widetilde{SL}_2(\mathbb{R})$. The center is $Z(\widetilde{\Gamma}) = C$ because $\Gamma_{p,q}$ has no center. We have obtained a presentation

$$\widetilde{\Gamma} \cong \langle \alpha, \beta ; \alpha^p = \beta^q \rangle \quad (22)$$

which is exactly the presentation (11) of the torus knot group. \square

4.4 Properties of the discrete group $\langle \alpha, \beta ; \alpha^p = \beta^q \rangle$

In this section we discuss some properties of the group $\widetilde{\Gamma}$ defined by the above presentation. Most of these properties, if not all, are well-known, but we shall sketch the proofs for completeness and to be able to give the formulations suitable for our purposes. In particular, we shall see that for any natural number r co-prime with p and q , there is a normal subgroup $G_r \subset \widetilde{\Gamma}$ of index r , isomorphic to $\widetilde{\Gamma}$ as an abstract group. Although many of the properties of $\widetilde{\Gamma}$ can be deduced from the presentation, we shall use the notation and some facts resulting from lemma 4.3. In particular, the center $Z(\widetilde{\Gamma})$ coincides with the center of $\widetilde{SL}_2(\mathbb{R})$, i.e.

$$Z(\widetilde{\Gamma}) = \langle c \rangle, \quad c = \alpha^p = \beta^q.$$

We have $\widetilde{\Gamma}/Z(\widetilde{\Gamma}) \cong \Gamma_{p,q}$.

Let us set some notational conventions. For $g \in \widetilde{\Gamma}$ we denote by $\bar{g} := g\widetilde{\Gamma}'$ and $g_0 = gZ(\widetilde{\Gamma})$ the corresponding cosets with respect to the commutator subgroup and the center of $\widetilde{\Gamma}$. Also, we let $N_{\widetilde{\Gamma}}(g)$ denote the smallest normal subgroup of $\widetilde{\Gamma}$ containing g (and similarly for other groups instead of $\widetilde{\Gamma}$).

Proposition 4.4. (i) *The factorgroup $H := \widetilde{\Gamma}/\widetilde{\Gamma}'$ is infinite cyclic, generated by the projection \bar{x} of the element $x := \alpha^{q_1}\beta^{p_1}$, where $pp_1 + qq_1 = 1$.*

(ii) *The canonical projection $\widetilde{\Gamma} \rightarrow \widetilde{\Gamma}/Z(\widetilde{\Gamma}) \cong \Gamma_{p,q}$ induces an isomorphism of the commutator subgroups $\widetilde{\Gamma}' \cong \Gamma'_{p,q}$.*

(iii) *We have $N_{\widetilde{\Gamma}}(x) = \widetilde{\Gamma}$, in other words, the group $\widetilde{\Gamma}$ is completely destroyed by adding the relation $x = 1$.*

Proof. From the presentation (11) one can see that the elements of $\widetilde{\Gamma}/\widetilde{\Gamma}'$ are of the form $\bar{\alpha}^i\bar{\beta}^j$. The equalities $\bar{x}^p = \bar{\beta}$ and $\bar{x}^q = \bar{\alpha}$ imply that \bar{x} generates $\widetilde{\Gamma}/\widetilde{\Gamma}'$. To see that $\widetilde{\Gamma}/\widetilde{\Gamma}'$ is

infinite, notice that it can be obtained from the free abelian group of rank two by adding only one relation $\bar{\alpha}^p \bar{\beta}^{-q} = 1$. This proves (i).

To prove (ii), notice that any surjective homomorphism of groups induces a surjective homomorphism of the commutator subgroups. We have to prove injectivity. Let us take $g \in \tilde{\Gamma}'$ and suppose $g_0 = 1$ in $\tilde{\Gamma}/Z(\tilde{\Gamma})$. This means $g = c^m$ for some $m \in \mathbb{Z}$. Hence $\bar{g} = \bar{x}^{pwm}$. On the other hand, $g \in \tilde{\Gamma}'$ implies $\bar{g} = 1$. Hence $m = 0$ and $g = 1$.

To prove (iii), in view of (i), it suffices to show that $N_{\tilde{\Gamma}}(x) \supset \tilde{\Gamma}'$. We have $\bar{x}^{pq} = \bar{c}$, so we can work in $\Gamma_{p,q}$ and show that $N_{\Gamma_{p,q}}(x_0) \supset \Gamma'_{p,q}$. Notice that $\Gamma'_{p,q} = N_{\Gamma_{p,q}}([\alpha_0^{q_1}, \beta_0^{p_1}])$ since q_1 is co-prime with p and p_1 is co-prime with q . Now, we have $[\alpha_0^{q_1}, \beta_0^{p_1}] = x_0 \alpha_0^{-q_1} x_0^{-1} \alpha_0^{q_1} \in N_{\Gamma_{p,q}}(x_0)$ and hence $N_{\Gamma_{p,q}}(x_0) \supset \Gamma'_{p,q}$. \square

From proposition 4.4(ii) and proposition 4.2 we obtain the following classical result.

Corollary 4.5. *The commutator subgroup $\tilde{\Gamma}'$, of the group $\tilde{\Gamma}$, is free of rank $(p-1)(q-1)$. Any of the following two sets of elements generate $\tilde{\Gamma}'$.*

$$\xi_{(i,j)} := [\alpha^i, \beta^j] = \alpha^i \beta^j \alpha^{-i} \beta^{-j} \quad , \quad i = 1, \dots, p-1 , j = 1, \dots, q-1 , \quad (23)$$

$$\eta_{(k,j)} := [\alpha, \alpha^k \beta^j \alpha^{-k}] = \alpha^k [\alpha, \beta^j] \alpha^{-k} \quad , \quad k = 0, \dots, p-2 , j = 1, \dots, q-1 . \quad (24)$$

It is a well-known fact that knot groups do not admit elements of finite order. In our case this is immediately deduced by the fact that $\tilde{\Gamma}'$ and $\tilde{\Gamma}/\tilde{\Gamma}'$ are free. It also follows from the presence of the faithful representation $\tilde{\Gamma} \longrightarrow \widetilde{SL}_2(\mathbb{R})$ obtained in lemma 4.3, and the fact that $\widetilde{SL}_2(\mathbb{R})$ has no elements of finite order.

Now we identify a family of subgroups G_r of $\tilde{\Gamma}$ such that each G_r is isomorphic to $\tilde{\Gamma}$, parametrized by natural numbers r co-prime to both p and q . Fix one such r and let r_p, r_q, p_r, q_r be integers such that

$$pp_r + rr_p = 1 \quad , \quad qq_r + rr_q = 1 .$$

Lemma 4.6. *Let us denote $\alpha_r := \alpha^r$, $\beta_r := \beta^r$, and let G_r be the subgroup of $\tilde{\Gamma}$ generated by α_r and β_r . Then G_r is isomorphic to $\tilde{\Gamma}$, and has the following properties.*

- (i) *The center $Z(G_r)$ is contained in $Z(\tilde{\Gamma})$ and generated by $c_r := c^r$.*
- (ii) *The commutator subgroup G'_r of G_r coincides with the commutator subgroup $\tilde{\Gamma}'$ of $\tilde{\Gamma}$.*
- (iii) *The factorgroup $H_r := G_r/G'_r$ is generated by the projection \bar{x}_r of the element $x_r := \alpha_r^{q_1} \beta_r^{p_1}$, where as earlier $pp_1 + qq_1 = 1$. We have $N_{\tilde{\Gamma}}(x_r) = N_{G_r}(x_r) = G_r$.*
- (iv) *G_r is a normal subgroup of $\tilde{\Gamma}$ and the factorgroup is cyclic of order r . There are isomorphisms*

$$\tilde{\Gamma}/G_r \cong H_r/H_r \cong Z(\tilde{\Gamma})/Z(G_r) \cong \mathbb{Z}_r .$$

- (v) *The restriction of the projection $\tilde{P} : \tilde{\Gamma} \longrightarrow \tilde{\Gamma}/Z(\tilde{\Gamma})$ to G_r is surjective, i.e.*

$$\tilde{P}(G_r) = \Gamma_{p,q} .$$

Proof. We obviously have $\alpha_r^p = \beta_r^q = c^r = c_r$. The group $\tilde{\Gamma}$ can be obtained as a free product with amalgamation of the free groups $\langle \alpha \rangle, \langle \beta \rangle$, with the amalgamated subgroups $\langle \alpha^p \rangle \sim \langle \beta^q \rangle$. The cyclic groups $\langle \alpha_r \rangle$ and $\langle \beta_r \rangle$ are subgroups of $\langle \alpha \rangle$ and $\langle \beta \rangle$ respectively. We have $\langle \alpha_r \rangle \cap \langle \alpha^p \rangle = \langle \alpha_r^p \rangle$ and $\langle \beta_r \rangle \cap \langle \beta^q \rangle = \langle \beta_r^q \rangle$. Thus

$$G_r \cong \langle \alpha_r, \beta_r ; \alpha_r^p = \beta_r^q \rangle .$$

Hence G_1 is isomorphic to $\tilde{\Gamma}$ and an isomorphism is given by

$$\mathcal{I}_r : \tilde{\Gamma} \longrightarrow G_r , \quad \alpha \longmapsto \alpha_r , \quad \beta \longmapsto \beta_r . \quad (25)$$

Since $c_r = \alpha_r^p = \beta_r^q$, property (i) is obvious. Moreover, we have $Z(\tilde{\Gamma})/Z(G_r) \cong \mathbb{Z}_r$.

To proof property (ii) it is sufficient to see that the elements $\xi_{(i,j)} = [\alpha^i, \beta^j]$ belong to G'_r (see corollary 4.5). We have

$$\alpha_r^{rp} = \alpha^{rrp} = \alpha^{rrp+ppr} c^{-pr} = \alpha c^{-pr} , \quad \beta_r^{rq} = \beta c^{-qr} .$$

Hence $[\alpha^i, \beta^j] = [\alpha_r^{irp}, \beta_r^{jrq}] \in G'_r$.

The proof of property (iii) is completely analogous to the proof of parts (i) and (iii) of proposition 4.4.

To prove (iv) we observe that, since $G'_r = \tilde{\Gamma}'$, there is a natural monomorphism

$$\begin{array}{ccc} H_r & \hookrightarrow & H \\ \bar{x}_r & \longmapsto & \bar{x}^r \end{array} .$$

The quotient is $H/H_r \cong \mathbb{Z}_r$.

The subgroup G_r is normal in $\tilde{\Gamma}$, because $\tilde{\Gamma}' = G'_r \subset G_r$ and for any pair of elements $g \in \tilde{\Gamma}$ and $g_1 \in G_r$ we have $gg_1g^{-1} = [g, g_1]g_1 \in G_r$. We obtain

$$\tilde{\Gamma}/G_r \cong (\tilde{\Gamma}/\tilde{\Gamma}')/(G_r/G'_r) \cong (\langle \bar{x} \rangle / \langle \bar{x}_r \rangle) \cong (\langle \bar{x} \rangle / \langle \bar{x}^r \rangle) \cong \mathbb{Z}_r .$$

It remains to prove (v). Recall that $\tilde{P}(\tilde{\Gamma}) = \Gamma_{p,q} = \langle \alpha_0, \beta_0 ; \alpha_0^p = \beta_0^q = 1 \rangle \cong \mathbb{Z}_p * \mathbb{Z}_q$, where $\alpha_0 = \tilde{P}(\alpha)$ and $\beta_0 = \tilde{P}(\beta)$. We have $\tilde{P}(\alpha_r)^{rp} = \alpha_0$ and $\tilde{P}(\beta_r)^{rq} = \beta_0$. Hence $\tilde{P}(G_r) = \Gamma_{p,q}$. The lemma is proved. \square

Corollary 4.7. *For any natural number r co-prime to both p and q , the coset space $\widetilde{SL}_2(\mathbb{R})/G_r$ is weakly homotopically equivalent to the torus knot complement $\mathbb{S}^3 \setminus K_{p,q}$.*

Our task now becomes to identify an r for which $\widetilde{SL}_2(\mathbb{R})/G_r$ is in fact diffeomorphic to the torus knot complement. As pointed out in the introduction, this has been done by Raymond and Vasquez, in [12], using the theory of Seifert fibrations. We intend to find an explicit diffeomorphism using the results on automorphic forms proved in the next section.

4.5 Automorphic forms for a (p, q, ∞) -triangle group

In this section we study automorphic forms related to the (p, q, ∞) -triangle group

$$\Gamma_{p,q} = \langle \alpha_0, \beta_0 ; \alpha_0^p = 1, \beta_0^q = 1 \rangle .$$

We shall refer to the fundamental domain D_1 for the action of $\Gamma_{p,q}$ on \mathbb{H}^2 described in remark 4.1. Recall that the elliptic element $\gamma_0 = (\alpha_0\beta_0)^{-1}$ generates the stabilizer of the cusp ∞ . According to formulae 17, we have

$$\gamma_0(z) = z + \lambda \quad , \quad \text{where} \quad \lambda = 2\left(\cos \frac{\pi}{p} + \cos \frac{\pi}{q}\right) .$$

The preimage $\widetilde{\Gamma} = \widetilde{P}^{-1}(\Gamma_{p,q})$ in $\widetilde{SL}_2(\mathbb{R})$ is generated by the preimages α, β as defined in the proof of lemma 4.3. We want to understand the algebra of automorphic forms $\mathcal{A}_{\widetilde{\Gamma}}^{*,*}$. Let us interpret the definition of automorphic forms for the group in question. Let $\omega(z, dz) = f(z)dz^k$ be a k -form on \mathbb{H}^2 , with $k \in \mathbb{Q}$ and $f(z)$ holomorphic on \mathbb{H}^2 . Let $\chi \in \text{Hom}(\widetilde{\Gamma}, U(1))$ be a character. Then ω is χ -automorphic if

- (a) $f(\alpha_0(z))\alpha'(z) = \chi(\alpha)f(z)$.
- (b) $f(\beta_0(z))\beta'(z) = \chi(\beta)f(z)$.
- (c) $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i \frac{z}{\lambda}}$.

Remark 4.4. Notice that a character χ of $\widetilde{\Gamma}$ is completely determined by its value on the element $x = \alpha^{q_1}\beta^{p_1}$. This follows directly from the facts that the kernel of any character must contain the commutator subgroup $\widetilde{\Gamma}'$ and that \overline{x} generates $\widetilde{\Gamma}/\widetilde{\Gamma}'$.

Lemma 4.8. Let $\omega(z, dz) = f(z)dz^k$ be a nonzero χ -automorphic k -form for $\widetilde{\Gamma}$. Let n_a , n_b and n_{∞} denote the orders of vanishing of f at a , b and ∞ respectively; let N denote the total number of other zeros of f in the fundamental domain D_1 , counted with multiplicities, with the identifications of the edges taken into account. Then

$$N + n_{\infty} + \frac{n_a}{p} + \frac{n_b}{q} = k \frac{pq - p - q}{pq} .$$

The proof is standard, uses a contour integral, and is analogous to the proof of lemma 1 on page I-14 of [11].

Corollary 4.9. In the notation of the above lemma, the number $m = k(pq - p - q)$ is an integer. The integer m is divisible by p (respectively, by q) if and only if n_a (respectively, n_b) is. In particular, if $n_a = n_b = 0$, then $k \frac{pq - p - q}{pq}$ is an integer.

Put

$$k_o = \frac{pq}{pq - p - q} .$$

(This number has some geometric meaning. Namely, the fundamental triangle Δ (see section 4.1) has Lobachevski area $\frac{\pi}{k_o}$. This follows directly from the formula $\frac{1}{k_o} = 1 - \frac{1}{p} - \frac{1}{q}$.)

The following two lemmas are taken from Milnor's article [9]. They relate the character of an automorphic form to the order of vanishing of the form at the vertices a and b .

Lemma 4.10. *In the notation of lemma 4.8, the values of the character χ on the generators α and β of $\tilde{\Gamma}$ are*

$$\chi(\alpha) = e^{2\pi i(\frac{k+n_a}{p})} \quad , \quad \chi(\beta) = e^{2\pi i(\frac{k+n_b}{q})} .$$

In particular, the value of χ on the element x is $\chi(x) = e^{2\pi i \frac{k}{pq}} e^{2\pi i \frac{qq_1 n_a + pp_1 n_b}{pq}}$.

Proof. We will evaluate $\chi(\alpha)$, the case of β being analogous. Since ω is χ -automorphic, f satisfies the relation

$$f(\alpha(z))(\alpha'(z))^k = \chi(\alpha)f(z) .$$

Notice that $(\alpha'(a))^k$ projects to $e^{2\pi i \frac{k}{p}}$ in \mathbb{C}^\times . Expand $f(z)$ and $f(\alpha(z))$ as Taylor series about a :

$$f(z) = \sum_{n=n_a}^{\infty} f_n(z-a)^n \quad , \quad f(\alpha(z)) = \sum_{n=n_a}^{\infty} f_n e^{2\pi i \frac{k}{p}} (z-a)^n .$$

Now the automorphy relation can be written as

$$\sum_{n=n_a}^{\infty} e^{2\pi i \frac{k}{p}} f_n e^{2\pi i \frac{n}{p}} (z-a)^n = \sum_{n=n_a}^{\infty} \chi(\alpha) f_n (z-a)^n .$$

A comparison of the the coefficients in front of $(z-a)^{n_a}$ yields the desired

$$e^{2\pi i \frac{n_a+k}{p}} = \chi(\alpha) .$$

The last statement in the lemma follows immediately from the expression $x = \alpha^{q_1} \beta^{p_1}$. \square

Now, we identify a character χ_o , which is of some particular importance. Set

$$\chi_o(x) = e^{2\pi i \frac{k_o}{pq}} = e^{2\pi i \frac{1}{pq-p-q}} .$$

On the generators α and β we get

$$\chi_o(\alpha) = e^{2\pi i \frac{k_o}{p}} = e^{2\pi i \frac{q}{pq-p-q}} \quad , \quad \chi_o(\beta) = e^{2\pi i \frac{k_o}{q}} = e^{2\pi i \frac{q}{pq-p-q}} .$$

Clearly, we have $\chi_o^{pq-p-q} = 1$.

Lemma 4.11. *In the notation of lemma 4.8, assume $n_a = n_b = 0$. Then k/k_o is a non-negative integer and $\chi = \chi_o^{k/k_o}$.*

Proof. The fact that k/k_o is a non-negative integer follows directly from lemma 4.8. From lemma 4.10 we obtain $\chi(\alpha) = e^{2\pi i \frac{k}{p}}$ and $\chi(\beta) = e^{2\pi i \frac{k}{q}}$, whence

$$\chi(x) = e^{2\pi i \frac{k}{pq}} = e^{2\pi i \frac{k_o}{pq} \frac{k}{k_o}} = \chi_o(x)^{k/k_o} .$$

Since the characters of $\tilde{\Gamma}$ are determined by their values on x , we get $\chi = \chi^{k/k_o}$. \square

We now proceed to construct some specific automorphic forms. Recall that from proposition 4.1 that the orbifold Θ obtained as a quotient of \mathbb{H}^2 under the action of the triangle group $\Gamma_{p,q}$, has an underlying surface X_Θ isomorphic to \mathbb{C} . The singular locus Σ_Θ consists of two cone points of indices p and q respectively: the images of the vertices a and b of the fundamental domain D_1 . Thus we have a uniformizing function for Θ ,

$$\theta : \mathbb{H}^2 \longrightarrow \mathbb{C} = X_\Theta ,$$

which can be chosen so that $\theta(a) = 0$ with multiplicity p , $\theta(b) = 1$ with multiplicity q . Let $\widehat{\mathbb{H}^2} = \mathbb{H}^2 \cup \{\infty\}$ and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$. Then θ extends to a function $\theta : \widehat{\mathbb{H}^2} \longrightarrow \widehat{\mathbb{C}}$ with a simple pole at ∞ .

The function θ is invariant³ on the orbits of $\Gamma_{p,q}$, i.e. $\theta(\gamma(z)) = \theta(z)$ for all $\gamma \in \Gamma_{p,q}$. The derivative $\theta'(z)$ satisfies

$$\theta'(\gamma(z)) = (\gamma'(z))^{-1} \theta'(z) \quad , \quad \gamma \in \Gamma_{p,q} .$$

Set

$$\begin{aligned} f_a &= \left(\frac{(\theta')^q}{\theta(\theta-1)^{q-1}} \right)^{\frac{1}{pq-p-q}} , \\ f_b &= \left(\frac{(\theta')^p}{\theta^{p-1}(\theta-1)} \right)^{\frac{1}{pq-p-q}} , \\ f_\infty &= \left(\frac{(\theta')^{pq}}{\theta^{(p-1)q}(\theta-1)^{p(q-1)}} \right)^{\frac{1}{pq-p-q}} . \end{aligned}$$

It is a matter of direct verification to show that f_a, f_b, f_∞ are holomorphic functions on \mathbb{H}^2 with finite limits at ∞ . Furthermore, f_a (resp. f_b, f_∞) has a simple zero at a (resp. b, ∞) and nowhere else in the closure $\widehat{D}_1 = \overline{D}_1 \cup \{\infty\}$ of the fundamental domain of $\Gamma_{p,q}$ given in 4.1.

Lemma 4.12. *The forms*

$$\omega_a(z, dz) = f_a(z) dz^{\frac{q}{pq-p-q}} \quad , \quad \omega_b(z, dz) = f_b(z) dz^{\frac{p}{pq-p-q}} \quad , \quad \omega_\infty(z, dz) = f_\infty(z) dz^{\frac{pq}{pq-p-q}}$$

are automorphic forms for $\widetilde{\Gamma}$. The character of ω_∞ is χ_o . The characters χ_a and χ_b of ω_a and ω_b evaluated at x give

$$\chi_a(x) = e^{2\pi i \frac{1}{p(pq-p-q)}} e^{2\pi i \frac{q_1}{p}} \quad , \quad \chi_b(x) = e^{2\pi i \frac{1}{q(pq-p-q)}} e^{2\pi i \frac{p_1}{q}} .$$

The relations $\chi_a^p = \chi_b^q = \chi_o$ hold.

³In many texts such a function would be called automorphic, but it does not satisfy the definition adopted here because of the pole at ∞ .

Proof. For $\gamma \in \tilde{\Gamma}$ we have

$$\begin{aligned}
f_a(\gamma(z)) &= \left(\frac{(\theta'(\gamma(z)))^q}{\theta(\gamma(z))(\theta(\gamma(z)) - 1)^{q-1}} \right)^{\frac{1}{pq-p-q}} \\
&= \left(\frac{((\gamma'(z))^{-1}\theta'(z))^q}{\theta(z)(\theta(z) - 1)^{q-1}} \right)^{\frac{1}{pq-p-q}} \\
&= \chi_a(\gamma)(\gamma'(z))^{\frac{-q}{pq-p-q}} \left(\frac{(\theta'(z))^q}{\theta(z)(\theta(z) - 1)^{q-1}} \right)^{\frac{1}{pq-p-q}} \\
&= \chi_a(\gamma)(\gamma'(z))^{\frac{-q}{pq-p-q}} f_a(z)
\end{aligned}$$

where $\chi_a(\gamma)$ is a $(pq - p - q)$ -th root of 1. The mapping $\gamma \mapsto \chi_a(\gamma)$ defines a character, and ω is a χ_a -automorphic form. Since the zeros of f_a are known, lemma 4.10 yields

$$\chi_a(x) = e^{2\pi i \frac{1}{p(pq-p-q)}} e^{2\pi i \frac{q_1}{p}}.$$

The situation with ω_b and ω_∞ is completely analogous. The relation between the characters follows immediately from the explicit formulae. \square

Lemma 4.13. *The forms $\omega_a, \omega_b, \omega_\infty$ defined above satisfy a relation of the form*

$$\omega_\infty = c_a \omega_a^p + c_b \omega_b^q.$$

for some nonzero complex numbers c_a, c_b .

Proof. For $u, v \in \mathbb{C}$ we compute

$$\begin{aligned}
uf_a^p + vf_\infty &= u \left(\frac{(\theta')^{pq}}{\theta^p(\theta - 1)^{p(q-1)}} \right)^{\frac{1}{pq-p-q}} + v \left(\frac{(\theta')^{pq}}{\theta^{(p-1)q}(\theta - 1)^{p(q-1)}} \right)^{\frac{1}{pq-p-q}} \\
&= \left(\frac{(\theta')^{pq}}{\theta^{(p-1)q}(\theta - 1)^q} \right)^{\frac{1}{pq-p-q}} (u\varepsilon_1\theta(\theta - 1)^{-1} + v\varepsilon_2(\theta - 1)^{-1}) \\
&= f_b^q \left(\frac{u\varepsilon_1\theta + v\varepsilon_2}{\theta - 1} \right)
\end{aligned}$$

where ε_1 and ε_2 are suitable roots of unity of order $pq - p - q$. Hence for $u = \varepsilon_1^{-1}$ and $v = -\varepsilon_2^{-1}$ we get $uf_a^p + vf_\infty = f_b^q$ which implies the statement of the lemma. \square

Theorem 4.14. *The forms ω_a and ω_b generate the algebra $\mathcal{A}_{\tilde{\Gamma}}^{*,*}$.*

Proof. Let $\omega(z, dz) = f(z)dz^k \in \mathcal{A}_{\tilde{\Gamma}}^{k,\chi}$ be a nonconstant automorphic form. In view of lemma 4.13, it is sufficient to express f as a polynomial in f_a, f_b, f_∞ . We shall use the notation from lemma 4.8 for the orders of vanishing of f . According to corollary 4.9, the number $m = k(pq - p - q)$ is an integer, which is divisible by p if and only if n_a is. We shall consider two cases.

Case 1: Suppose that p divides m , say $m = ps$. Then $n_1 = \frac{n_a}{p}$ is also an integer. Then the form ω_b^s is a k -form. Moreover, we have $\chi = \chi_b^s$. To see that we just evaluate the two character at x using lemma 4.10:

$$\begin{aligned}\chi(x) &= e^{2\pi i \frac{m}{pq(pq-p-q)}} e^{2\pi i \frac{qq_1 n_a + pp_1 n_b}{pq}} = e^{2\pi i \frac{s}{q(pq-p-q)}} e^{2\pi i \frac{qq_1 p n_1 + pp_1 n_b}{pq}} = e^{2\pi i \frac{s}{q(pq-p-q)}} e^{2\pi i \frac{p_1 n_b}{q}}, \\ \chi_b^s(x) &= e^{2\pi i \frac{s}{q(pq-p-q)}} e^{2\pi i \frac{p_1 s}{q}}.\end{aligned}$$

Thus $\chi = \chi_b^s$ is equivalent to $e^{2\pi i \frac{p_1 n_b}{q}} = e^{2\pi i \frac{p_1 s}{q}}$, which in turn holds if and only if q divides $s - n_b$. The identity proven in lemma 4.8, properly rewritten according to the present assumptions, becomes

$$ps = pqN + pqn_\infty + qp n_1 + pn_b.$$

The above equality implies that q divides $s - n_b$. Hence $\chi = \chi_b^s$. We can conclude that both ω and ω_b^s belong to the vector space $\mathcal{A}_{\tilde{\Gamma}}^{k,\chi}$. Hence any linear combination of these two forms belongs to $\mathcal{A}_{\tilde{\Gamma}}^{k,\chi}$. There exists $c_1 \in \mathbb{C}$ such that the form $f - c_1 f_b$ vanishes at ∞ and can be written as

$$f - c_1 f_b^s = f_\infty f_1, \quad f_1(z) dz^{k_1} \in \mathcal{A}_{\tilde{\Gamma}}^{k_1, \chi_1},$$

where $\chi_1 = \chi \chi_o^{-1}$ and $k_1 = k - k_o = \frac{p(s-q)}{pq-p-q}$. Put $m_1 = p(s-q)$. Now notice the following: the fact that $\mathcal{A}_{\tilde{\Gamma}}^{k,\chi}$ contains cusp forms implies, via lemma 4.8, that $m \geq pq$. Thus $s \geq q$. There are two possible outcomes, depending on whether $s = q$ or $s > q$. First, if $s = q$, then $k_1 = 0$, so that f_1 is a holomorphic function on $\widehat{\mathbb{H}^2/\Gamma_{p,q}}$ and hence constant. In this case $f = c_1 f_b^s + f_1 f_\infty$, and we are done. Second, if $s > q$, then p divides m_1 and we can restart the procedure. Eventually f is expressed as a polynomial in f_b and f_∞ .

Case 2: Suppose now that p does not divide m . Then $n_a > 0$. Consider $f_1 = \frac{f}{f_a^{n_a}}$. Put $m_1 = m - qn_a$. Then $f_1(z) dz^{\frac{m_1}{pq-p-q}}$ is $\chi \chi_a^{-n_a}$ -automorphic. We have $f_1(a) \neq 0$ and hence p divides m_1 , which brings us to case 1.

This completes the proof of the theorem. \square

Corollary 4.15. *Let $G = G_{pq-p-q}$ be the subgroup of $\tilde{\Gamma}$ generated by α^{pq-p-q} and β^{pq-p-q} . Then $\mathcal{A}_G^* = \mathcal{A}_{\tilde{\Gamma}}^{*,*}$, and hence the algebra \mathcal{A}_G^* is generated by ω_a and ω_b .*

Proof. To prove the corollary it suffices to show that G equals the kernel of each of the characters χ_a and χ_b . According to lemma 4.6 the group G is isomorphic to $\tilde{\Gamma}$ as an abstract group, and equals the smallest normal subgroup of $\tilde{\Gamma}$ containing the element x^{pq-p-q} . Also, we have $\tilde{P}(G) = \Gamma_{p,q}$. From lemma 4.12 we know that $\chi_a^p = \chi_b^q = \chi_o$. We also know that $\chi_o(x) = e^{2\pi i \frac{1}{pq-p-q}}$ so that χ_o has order $pq - p - q$. Let s be the order of χ_a . Then $1 = \chi_a^{ps} = \chi_o^s$, whence $pq - p - q$ divides s . On the other hand, we have

$$\begin{aligned}\chi_a(x) &= e^{2\pi i \frac{1}{p(pq-p-q)}} e^{2\pi i \frac{q_1}{p}} = e^{2\pi i \frac{1 + pq q_1 - pq_1 - qq_1}{p(pq-p-q)}} = e^{2\pi i \frac{pp_1 + qq_1 + pq q_1 - pq_1 - qq_1}{p(pq-p-q)}} \\ &= e^{2\pi i \frac{pp_1 + pq q_1 - pq_1}{p(pq-p-q)}} = e^{2\pi i \frac{p_1 + qq_1 - q_1}{pq-p-q}}.\end{aligned}$$

Hence $\chi_a^{pq-p-q} = 1$ and so s divides $pq - p - q$. Thus $s = pq - p - q$. Analogous argument shows that the order of χ_b equals $pq - p - q$. This completes the proof. \square

5 The universal covering $\widetilde{SL}_2(\mathbb{R}) \rightarrow \mathbb{S}^3 \setminus K_{p,q}$

In this section we show that \widetilde{SL}_2/G is diffeomorphic to the complement $\mathbb{S}^3 \setminus K_{p,q}$ of a torus knot in the 3-sphere. Consider the map

$$\begin{aligned} \Psi : \mathbb{H}^2 \times \widetilde{\mathbb{C}^\times} &\longrightarrow \mathbb{C}^2 \\ (z, w) &\longmapsto (\omega_b(z, w), \omega_a(z, w)) \end{aligned} .$$

Corollary 4.15 asserts that the map Ψ factors through the projection $P_G : \mathbb{H}^2 \times \widetilde{\mathbb{C}^\times} \rightarrow (\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times})/G$. Put $M = (\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times})/G$ and denote the resulting map by

$$\overline{\Psi} : M \longrightarrow \mathbb{C}^2 .$$

Now recall that we have $\omega_\infty = c_b \omega_b^q + c_a \omega_a^p$. The cusp form ω_∞ does not vanish on $\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$. It follows that the image of $\overline{\Psi}$ sits in the complement of the curve $V \subset \mathbb{C}^2$ defined by the equation $c_b z_1^q + c_a z_2^p = 0$.

Theorem 5.1. *The map $\overline{\Psi} : M \rightarrow \mathbb{C}^2 \setminus V$ is biholomorphic.*

Proof. We shall prove that $\overline{\Psi}$ is a bijection. Since ω_a and ω_b are holomorphic functions on M , this will be sufficient to conclude that $\overline{\Psi}$ is biholomorphic (see [2], p. 179). Put $r = pq - p - q$.

First we prove the injectivity of $\overline{\Psi}$. Recall that ω_a and ω_b generate \mathcal{A}_G^* . Hence if $\Psi(z, w) = \Psi(z_0, w_0)$, then $\omega(z, w) = \omega(z_0, w_0)$ for all $\omega \in \mathcal{A}_G^*$. To prove injectivity it is sufficient to show that for any two points $(z, w), (z_0, w_0) \in \mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$ which are not congruent under the action of G , there exists a G -automorphic form ω such that $\omega(z_0, w_0) \neq \omega(z, w)$. We can assume that both z and z_0 belong to the fundamental domain D_1 of $\Gamma_{p,q}$. Recall that ω_a^p and ω_b^q belong to $\mathcal{A}_G^{k_o}$. Also, f_a^p vanishes at the vertex a with multiplicity p and has no other zeros in D_1 . Similarly, f_b^q vanishes at b and nowhere else in $\overline{D_1}$. Thus there exists a unique (up to a scalar multiple) linear combination $f = c_1 f_a^p + c_2 f_b^q$ which vanishes at z_0 , and this f does not vanish at point in $\overline{D_1}$ not congruent to z_0 . Now the point z is either congruent to z_0 , or not. If z is not congruent to z_0 , then $\omega(z, dz) = f(z) dz^{k_o}$ vanishes at (z_0, w_0) but not at (z, w) . If z is congruent to z_0 , we can assume that $z = z_0$ and that $\omega(z, dz)$ does not vanish at (z_0, w_0) nor at (z_0, w) . However, an elementary direct computation shows that, if $\Psi(z_0, w_0) = \Psi(z_0, w)$, then $w_0 = w(c'(z_0))^{jr}$ for some integer j , so that (z_0, w_0) and (z_0, w) are congruent under (the center of) G .

Now we prove the surjectivity of $\overline{\Psi}$. Start by noticing that both $\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$ and \mathbb{C}^2 carry $\widetilde{\mathbb{C}^\times}$ actions with respect to which Ψ is equivariant. These actions are given by

$$\begin{aligned} \widetilde{\mathbb{C}^\times} \times (\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}) &\longrightarrow \mathbb{H}^2 \times \widetilde{\mathbb{C}^\times} & \widetilde{\mathbb{C}^\times} \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ \lambda \cdot (z, w) &= (z, \lambda w) & \lambda \cdot (z_1, z_2) &= (\lambda^{\frac{p}{r}} z_1, \lambda^{\frac{q}{r}} z_2) \end{aligned} .$$

(Recall that $\widetilde{\mathbb{C}^\times}$ acts on \mathbb{C} via the projection to \mathbb{C}^\times .) The second action was already introduced in section 2.3. Let us recall and re-examine the orbit structure. It is easy to see that each orbit closure has the form

$$\{(z_1, z_2) \in \mathbb{C}^2 : c_1 z_1^q + c_2 z_2^p = 0\}$$

for some suitable pair of complex numbers (c_1, c_2) . Conversely, each pair of complex numbers (c_1, c_2) corresponds to an orbit closure. We want to show that all orbits, except the ones corresponding to pairs proportional to (c_b, c_a) , are contained in the image of Ψ . Let (c_1, c_2) be a pair not proportional to (c_b, c_a) . Set

$$\omega = c_1 \omega_b^q + c_2 \omega_a^p.$$

Then ω is a form in $\mathcal{A}_G^{k_o}$ and we have $\omega(z, dz) = f(z)dz^{k_o}$ with $f = c_1 f_b^q + c_2 f_a^p$. The choice of c_1, c_2 implies that f is not a scalar multiple of f_∞ . Hence f vanishes somewhere inside \mathbb{H}^2 , say $f(z_0) = 0$. Then $\omega(z_0, dz) = 0$ and we have

$$\Psi(\{z_0\} \times \widetilde{\mathbb{C}^\times}) = \{(z_1, z_2) \in \mathbb{C}^2 \setminus (0, 0) : c_1 z_1^q + c_2 z_2^p = 0\}.$$

Thus the image of Ψ equals $\mathbb{C}^2 \setminus V$ which implies the surjectivity of $\overline{\Psi}$. \square

Corollary 5.2. *The coset space $\widetilde{SL}_2(\mathbb{R})/G$ is diffeomorphic to the complement $\mathbb{S}^3 \setminus K$ of a (p, q) -torus knot in the 3-sphere.*

Proof. Recall that we have identified $\widetilde{SL}_2(\mathbb{R})$ with its orbit in $\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$ through the point $(i, 1)$. Restrict the two $\widetilde{\mathbb{C}^\times}$ actions from the above proof to \mathbb{R}_+ actions, i.e. consider only real positive λ . Then each \mathbb{R}_+ orbit in $\mathbb{H}^2 \times \widetilde{\mathbb{C}^\times}$ intersects each orbit of $\widetilde{SL}_2(\mathbb{R})$ at exactly one point. On the other hand, each \mathbb{R}_+ orbit in \mathbb{C}^2 intersects the three sphere $\mathbb{S}^3 = \{|z_1|^2 + |z_2|^2 = 1\}$ at exactly one point. The curve V and its complement $\mathbb{C}^2 \setminus V$ are invariant under the \mathbb{R}^+ action. The intersection $K = \mathbb{S}^3 \cap V$ is a (p, q) -torus knot. Thus each \mathbb{R}_+ orbit in $\mathbb{C}^2 \setminus V$ intersects both $\Psi(\widetilde{SL}_2(\mathbb{R}))$ and $\mathbb{S}^3 \setminus K$, each at a single point.

Define a map $\rho : \Psi(\widetilde{SL}_2(\mathbb{R})) \rightarrow \mathbb{S}^3 \setminus K$ sending a point of $\Psi(\widetilde{SL}_2(\mathbb{R}))$ to the unique point in $\mathbb{S}^3 \setminus K$ which belongs to the same \mathbb{R}_+ orbit as (z_1, z_2) . The \mathbb{R}_+ orbit through a point $(z_1, z_2) \in \Psi(\widetilde{SL}_2(\mathbb{R}))$ has the form $\{(\lambda^{\frac{p}{r}} z_1, \lambda^{\frac{q}{r}} z_2) : \lambda \in \mathbb{R}_+\}$. The function $F(\lambda, z_1, z_2) = \|(\lambda^{\frac{p}{r}} z_1, \lambda^{\frac{q}{r}} z_2)\|^2$ takes all positive values, and has nonvanishing partial derivative $\frac{\partial F}{\partial \lambda}$ at all points. The implicit function theorem implies that the value of λ for which $F(\lambda, z_1, z_2) = 1$ depends smoothly on (z_1, z_2) . We can conclude that the map ρ is a diffeomorphism. \square

Remark 5.1. *The above corollary shows that the torus knot complement is a homogeneous manifold, i.e. admits a transitive group action. On the other hand, $\widetilde{SL}_2(\mathbb{R})$ with its left invariant metric is a model for one of the eight 3-dimensional geometries, as discussed in section 2.1 and remark 3.1. This metric is not right invariant. The coset $\widetilde{SL}_2(\mathbb{R})/G$ is obtained by letting G act on $\widetilde{SL}_2(\mathbb{R})$ on the left. Thus the coset, and consequently $\mathbb{S}^3 \setminus K$,*

inherits a locally homogeneous metric. However, the latter metric is not homogeneous, i.e. $\mathbb{S}^3 \setminus K$ does not admit a transitive isometry group. There is only a 1-parameter isometric action on the torus knot complement, induced by the right action of the subgroup $\widetilde{SO}_2(\mathbb{R})$ on $\widetilde{SL}_2(\mathbb{R})$, and providing the Seifert fibration structure.

6 A knot in a lens space

We saw, in corollary 5.2, that the complement $\mathbb{S}^3 \setminus K_{p,q}$ of a (p,q) -torus knot in the 3-sphere is diffeomorphic to the coset space $\widetilde{SL}_2(\mathbb{R})/G$ where G is a certain subgroup of the preimage $\widetilde{\Gamma} = \widetilde{P}^{-1}(\Gamma_{p,q})$ of a (p,q,∞) -triangle group $\Gamma_{p,q} \subset PSL_2(\mathbb{R})$. The index of G in $\widetilde{\Gamma}$ is $r = pq - p - q$. The only case when $G = \widetilde{\Gamma}$ is $p = 2, q = 3$, which gives $\Gamma_{p,q} = PSL_2(\mathbb{Z})$. In this case we get $\mathbb{S}^3 \setminus K_{2,3} \cong PSL_2(\mathbb{R})/PSL_2(\mathbb{Z})$. It is natural to ask whether the coset space $\widetilde{SL}_2(\mathbb{R})/\widetilde{\Gamma} = PSL_2(\mathbb{R})/\Gamma_{p,q}$ can be identified with some other space, perhaps related to the knot complement $\mathbb{S}^3 \setminus K_{p,q}$. The answer is not hard to find, and we present it in this section.

In theorem 5.1 and its proof we have constructed a $\widetilde{\mathbb{C}}^\times$ -equivariant biholomorphic map $\overline{\Psi} : (\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/G \longrightarrow \mathbb{C}^2 \setminus V$. Recall that the action of the center C of $\widetilde{SL}_2(\mathbb{R})$ on $\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$ coincides with the action of the subgroup of $\widetilde{\mathbb{C}}^\times$ which is the preimage of $1 \in \mathbb{C}^\times$ under the universal covering map. Now observe that together G and C generate exactly $\widetilde{\Gamma}$, so that

$$((\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/G)/C = (\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/\widetilde{\Gamma}.$$

Moreover, since $C \cap G = \langle c^r \rangle$, the action of C on $(\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/G$ reduces to a cyclic action of order r . From the construction in the proof of theorem 5.1 it follows that this cyclic action is transmitted to an action on $\mathbb{C}^2 \setminus V$ generated by the linear transformation

$$h_{\frac{1}{r}} = \begin{pmatrix} e^{2\pi i \frac{p}{r}} & 0 \\ 0 & e^{2\pi i \frac{q}{r}} \end{pmatrix}.$$

Clearly the latter action (as well as the $\widetilde{\mathbb{C}}^\times$ -action) extends from $\mathbb{C}^2 \setminus V$ to \mathbb{C}^2 , the curve V being an orbit closure. The action of $\langle h_{\frac{1}{r}} \rangle$ on $\mathbb{C}^2 \setminus \mathbf{o}$ is free. The quotient $\Sigma = \mathbb{C}^2 / \langle h_{\frac{1}{r}} \rangle$ is a singular surface. Let $W \subset \Sigma$ denote the image of V , which is a singular curve in Σ . Clearly

$$(\mathbb{C}^2 \setminus V) / \langle h_{\frac{1}{r}} \rangle = \Sigma \setminus W.$$

Thus we have a $\widetilde{\mathbb{C}}^\times$ -equivariant biholomorphic map between $(\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/\widetilde{\Gamma}$ and $\Sigma \setminus W$. Now recall that we have identified $\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times$ with the universal cover of the bundle $T'\mathbb{H}^2$ of nonzero tangent vectors to \mathbb{H}^2 . Since $\widetilde{\Gamma} = \widetilde{P}^{-1}(\Gamma_{p,q})$, we have $(\mathbb{H}^2 \times \widetilde{\mathbb{C}}^\times)/\widetilde{\Gamma} \cong T'\mathbb{H}^2/\Gamma_{p,q}$. To summarize, we have obtained the following.

Theorem 6.1. *The quotient $T'\mathbb{H}^2/\Gamma_{p,q}$ is \mathbb{C}^\times -equivariantly biholomorphic to the complex surface $\Sigma \setminus W$.*

Now let us turn our attention to the 3-manifolds. Recall that $PSL_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $U\mathbb{H}^2$. Hence $PSL_2(\mathbb{R})/\Gamma_{p,q} \cong U\mathbb{H}^2/\Gamma_{p,q} \subset T'\mathbb{H}^2/\Gamma_{p,q}$. The above theorem implies that $PSL_2(\mathbb{R})/\Gamma_{p,q}$ embeds in $\Sigma \setminus W$ in a way that it intersects each \mathbb{R}_+ orbit exactly once. On the other hand, consider the image of $\mathbb{S}^3 \setminus K$ in $\Sigma \setminus W$. Let \mathcal{L} and \mathcal{K} denote respectively the images of \mathbb{S}^3 and K in Σ . Notice that \mathcal{L} is a lens space: the quotient of \mathbb{S}^3 under the free action of the cyclic group generated by $h_{\frac{r}{q}}$. In the notation of [13] we have $\mathcal{L} = L(r, p(q_1 - p_1 + pp_1))$. Clearly $\mathcal{K} = \mathcal{L} \cap W$ and $\mathcal{L} \setminus \mathcal{K}$ is the image of $\mathbb{S}^3 \setminus K$ in Σ . Observe that $\mathcal{L} \setminus \mathcal{K}$ intersects each \mathbb{R}_+ orbit in $\Sigma \setminus W$ exactly once. Thus we can use the \mathbb{R}_+ action to obtain a diffeomorphism between $\mathcal{L} \setminus \mathcal{K}$ and the image of $PSL_2(\mathbb{R})/\Gamma_{p,q}$, as in the proof of corollary 5.2. Let us record this fact.

Corollary 6.2. *The coset space $PSL_2(\mathbb{R})/\Gamma_{p,q}$ is diffeomorphic to the knot complement in a lens space $\mathcal{L} \setminus \mathcal{K}$.*

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